Riding Upon the Horse of True Mathematics: Tullio Levi-Civita and his Impact on Einstein's Theory of General Relativity

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IHES

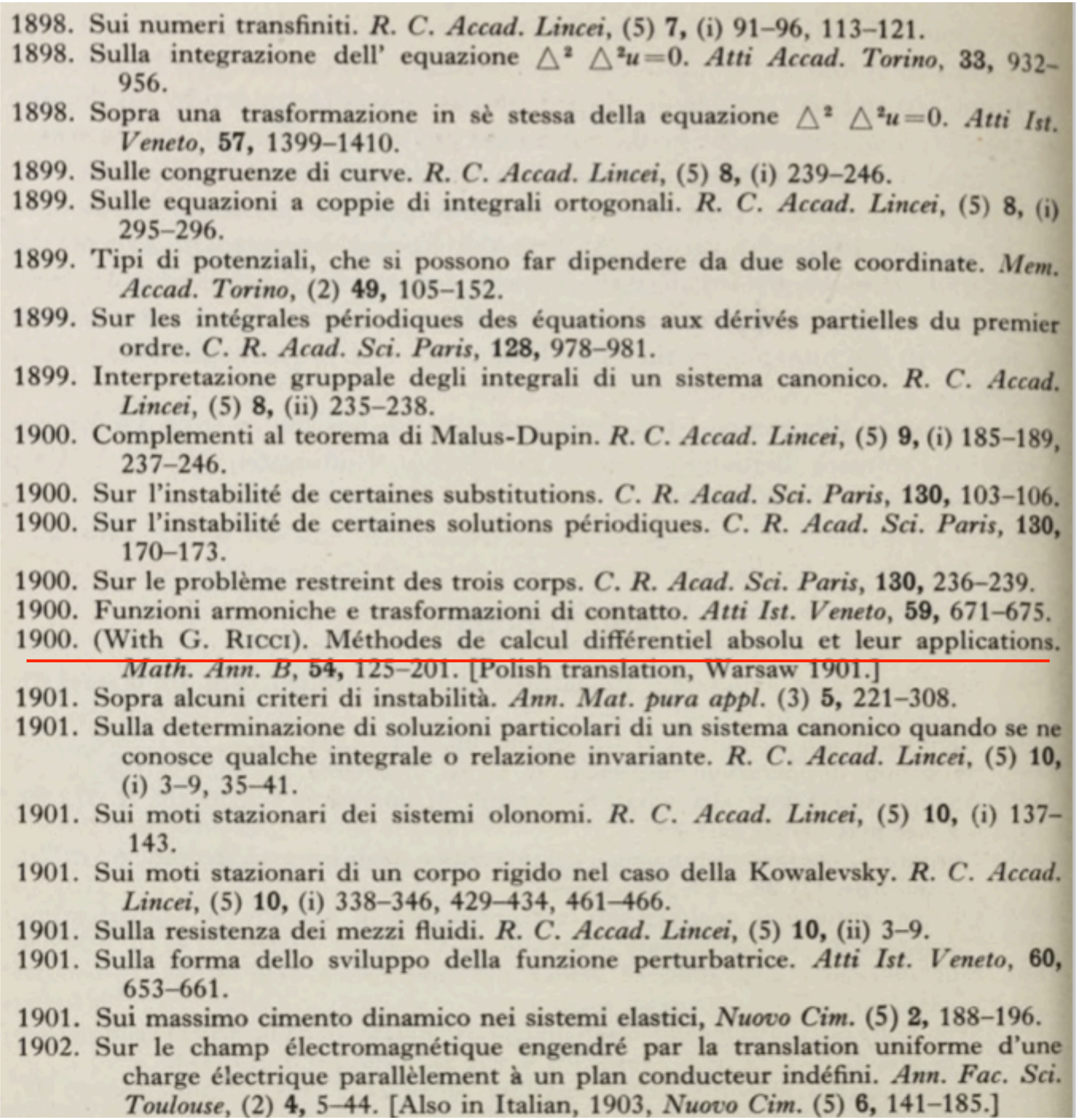
The legacy of Tullio Levi-Civita:

a scientific conference in honor of Tullio Levi-Civita

Padova, February 19 - 20, 2018

Levi-Civita's list of publications

A huge range of topics; surprisingly only a minority about the absolute differential calculus.
I expected more in view of the famous:
LC connection
LC tensor
LC metric



1898. Sui numeri transfiniti. *R. C. Accad. Lincei*, (5) **7,** (i) 91–96, 113–121.
1898. Sulla integrazione dell' equazione $\triangle^2 \triangle^2 u = 0$. *Atti Accad. Torino*, **33,** 932–956.
1898. Sopra una trasformazione in sè stessa della equazione $\triangle^2 \triangle^2 u = 0$. *Atti Ist. Veneto*, **57,** 1399–1410.
1899. Sulle congruenze di curve. *R. C. Accad. Lincei*, (5) **8,** (i) 239–246.
1899. Sulle equazioni a coppie di integrali ortogonali. *R. C. Accad. Lincei*, (5) **8,** (i) 295–296.
1899. Tipi di potenziali, che si possono far dipendere da due sole coordinate. *Mem. Accad. Torino*, (2) **49,** 105–152.
1899. Sur les intégrales périodiques des équations aux dérivés partielles du premier ordre. *C. R. Acad. Sci. Paris*, **128,** 978–981.
1899. Interpretazione gruppale degli integrali di un sistema canonico. *R. C. Accad. Lincei*, (5) **8,** (ii) 235–238.
1900. Complementi al teorema di Malus-Dupin. *R. C. Accad. Lincei*, (5) **9,** (i) 185–189, 237–246.
1900. Sur l'instabilité de certaines substitutions. *C. R. Acad. Sci. Paris*, **130,** 103–106.
1900. Sur l'instabilité de certaines solutions périodiques. *C. R. Acad. Sci. Paris*, **130,** 170–173.
1900. Sur le problème restreint des trois corps. *C. R. Acad. Sci. Paris*, **130,** 236–239.
1900. Funzioni armoniche e trasformazioni di contatto. *Atti Ist. Veneto*, **59,** 671–675.
1900. (With G. RICCI). Méthodes de calcul différentiel absolu et leur applications. *Math. Ann. B*, **54,** 125–201. [Polish translation, Warsaw 1901.]
1901. Sopra alcuni criteri di instabilità. *Ann. Mat. pura appl.* (3) **5,** 221–308.
1901. Sulla determinazione di soluzioni particolari di un sistema canonico quando se ne conosce qualche integrale o relazione invariante. *R. C. Accad. Lincei*, (5) **10,** (i) 3–9, 35–41.
1901. Sui moti stazionari dei sistemi olonomi. *R. C. Accad. Lincei*, (5) **10,** (i) 137–143.
1901. Sui moti stazionari di un corpo rigido nel caso della Kowalevsky. *R. C. Accad. Lincei*, (5) **10,** (i) 338–346, 429–434, 461–466.
1901. Sulla resistenza dei mezzi fluidi. *R. C. Accad. Lincei*, (5) **10,** (ii) 3–9.
1901. Sulla forma dello sviluppo della funzione perturbatrice. *Atti Ist. Veneto*, **60,** 653–661.
1901. Sui massimo cimento dinamico nei sistemi elastici, *Nuovo Cim.* (5) **2,** 188–196.
1902. Sur le champ électromagnétique engendré par la translation uniforme d'une charge électrique parallèlement à un plan conducteur indéfini. *Ann. Fac. Sci. Toulouse*, (2) **4,** 5–44. [Also in Italian, 1903, *Nuovo Cim.* (5) **6,** 141–185.]

1902. Influenza di uno schermo conduttore sul campo elettromagnetico di una corrente alternativa parallela allo schermo. *R. C. Accad. Lincei*, (5) **11,** (i) 163–170, 191–198, 228–237; also *Nuovo. Cim.* (5) **3,** 442–455.

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1903. Condition du choc dans le problème restreint des trois corps. *C. R. Acad. Sci. Paris*, **135,** 221–223.

1903. Traiettorie singolari ed urti nel problema ristretto dei tre corpi. *Ann. Mat. pura appl.* (3) **9,** 1–32.

1903. Sur la singularité dont sont affectées, pour une vitesse nulle, les équations du mouvement d'un point frottant sur une surface. *Arch. Math. Phys.* (3*B*) **5,**

1904. Sopra la equazione di Kepler. *R. C. Accad. Lincei*, (5) **13,** (i) 260–268; also *Astr. Nachr.* no. 3956, 313–314.

1904. Sopra un problema di elettrostatica che interessa la construzione dei cavi. *R. C. Accad. Lincei*, (5) **13,** (i) 375–382; also *Nuovo Cim.* (5) **8,** 187–195.

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1905. Sulle funzioni di due o più variabili complesse. *R. C. Accad. Lincei*, (5) **14,** (ii) 492–499.

1906. Sur la résolution qualitative du problème restreint des trois corps. *Acta Math. Stockh.* **30,** 305–327. [Summary in *Verh.* 3 *Int. Math. Congr.* (Leipzig, 1905), 402–408.]

1906. Sulla contrazione delle vene liquide. *Atti Ist. Veneto*, **64,** 1465–1472.

1906. Sur la recherche des solutions particulières des systèmes différentiels et sur les mouvements stationnaires. *Math.–phys. Abh. Warschau*, **17,** 1–40.

1906. The mixed transformations of Lagrange's equations. *Nature, Lond.* **74,** 516.

1906. Sulla penetrazione dei proiettili nei mezzi solidi. *Atti Ist. Veneto*, **65,** 1149–1158.

1906. Ueber einer technische Aufgabe die in Beziehung zur konformen Abbildung steht. *Verh. Ges. dtsch. Naturf. Aerzt.* **77,** 20–21.

1907. Le idee di Enriques sui principi della meccanica. *Riv. Filos. Sci. Aff.* **9,** 337–346.

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1915. Sul problema piano dei tre corpi. *R. C. Accad. Lincei*, (5) **24,** (ii) 421–433, 485–501, 553–569.

1916. Sulla introduzione di rincoli olonomi nelle equazioni dinamiche di Hamilton. *Atti Ist. Veneto*, **75,** 387–395.

1916. Sopra due trasformazioni canoniche desunte dal moto parabolico. *R. C. Accad. Lincei*, (5) **25,** (i) 445–458.

1916. Sur la régularisation du problème des trois corps. *C. R. Acad. Sci. Paris*, **162,** 625–628.

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1917. Nozione di parallelismo in una varietà qualunque e conseguente specificazione della curvatura riemanniana. *R. C. Circ. Mat. Palermo*, **42,** 173–215.

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1917/1919. ds^2 einsteiniani in campi newtoniani [note 1–9]. *R. C. Accad. Lincei*, (5) **26,** (ii) 307–317; **27,** (i) 3–12; **27,** (ii) 183–191, 220–229, 240–248, 283–292, 344–351; **28,** (i) 3–13, 101–109.

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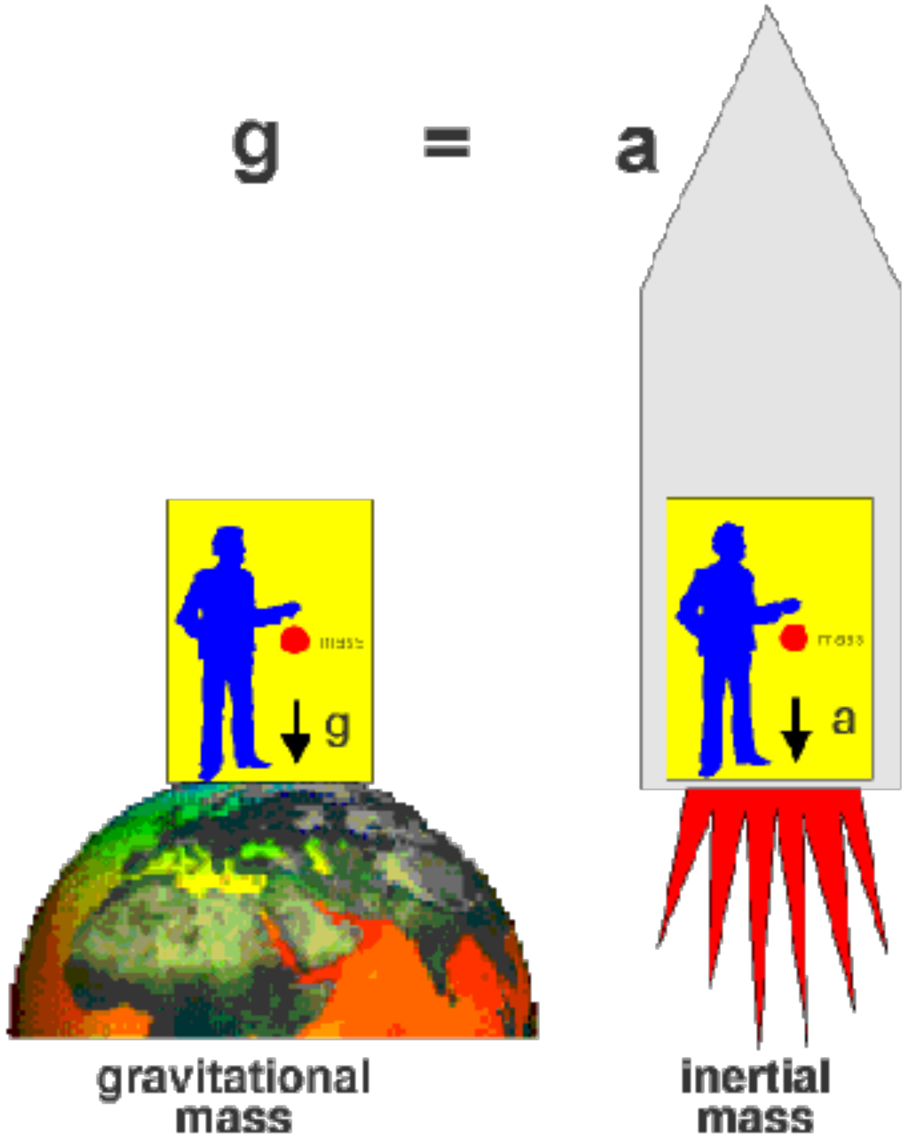
Einstein's path towards General Relativity



1905:Special Relativity
ds^2= — c^2 dt^2 + dx^2 + dy^2 +d z^2
1907 Einstein's Equivalence Principle (EP):
inertia = gravity
1912 « The speed of light and the statics of the gravitational field »
Considers a uniformly accelerated reference system (à la Born 1909) and derives (for the first time) the « Rindler metric »

$$ds^2 = -(c_0 + ax/c_0)^2\, dt^2 + dx^2 + dy^2 + dz^2$$

Suggests that $c(x) = c_0 + ax/c_0$ is exact for static, uniform gravitational field and that the spatial geometry stays Euclidean for this case.

1912 (same paper) Einstein mentions a uniformly rotating system and the fact that « π » is modified

$$\left[ds^2 = -(c^2 - \Omega^2(x^2+y^2))\, dt^2) + dx^2 + dy^2 + dz^2 - 2\Omega\, x\, dx\, dt + 2\Omega x\, dy\, dt\right]$$

Later in 1912 expects g_{oi} to be linked to motion of matter

Einstein's tools in building GR

All the forms of EP :

UFF → (wrongly) eliminates SR Scalar theory

inertia = gravitation → « Rindler metric » → rotating disk metric → $g_{\mu\nu}$

« Einstein's EP » → laws of non gravitational physics must be generally covariant in $g_{\mu\nu}$

$$\boxed{S = -mc \int \sqrt{-g_{\mu\nu}\, dx^\mu dx^\nu}}$$

« Strong EP » E=mc^2 → all forms of energy must contribute to mass, including gravitational energy

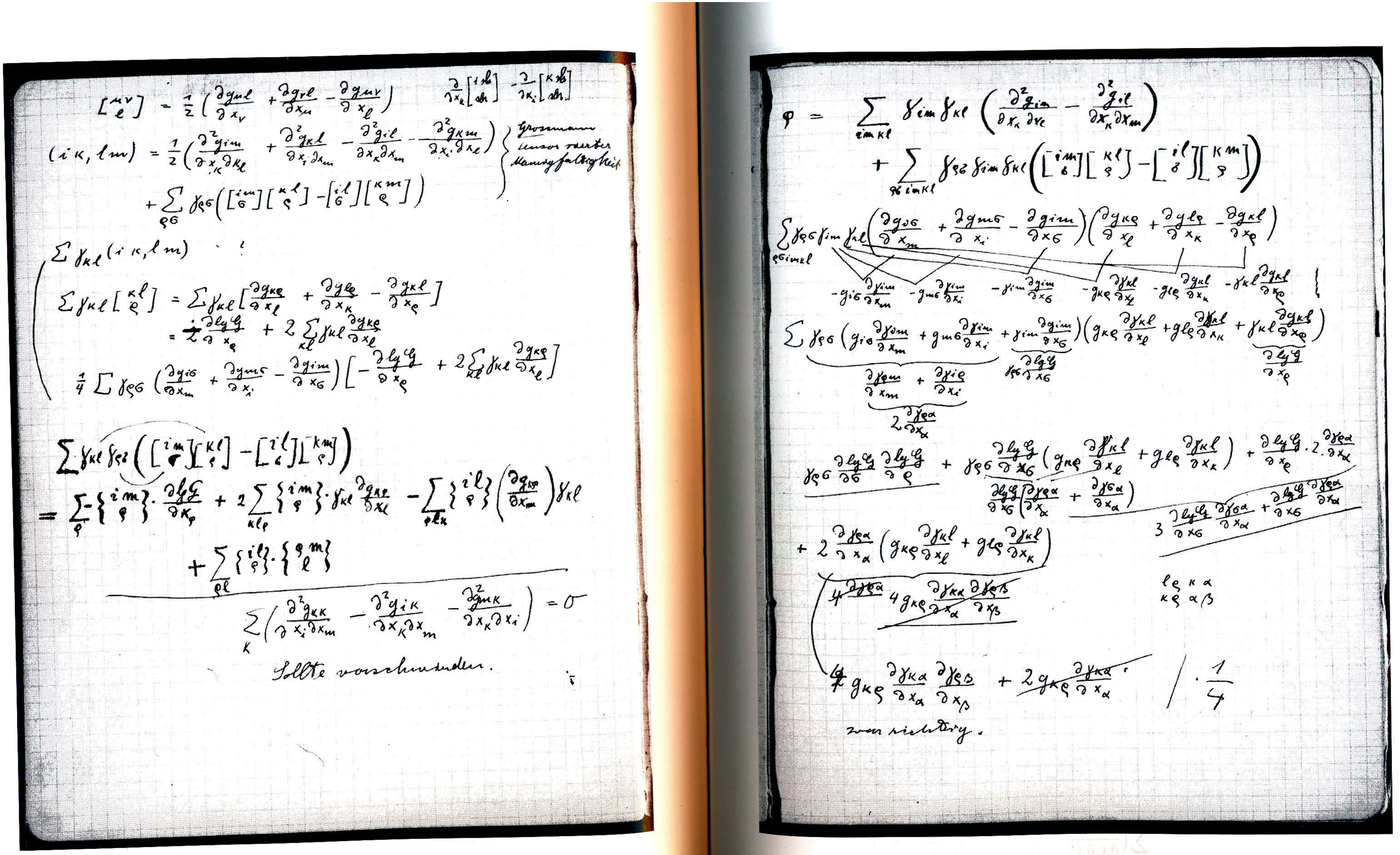
1912-1913 (Zurich notebook)

$$\boxed{\begin{aligned} O = \sqrt{g}\,\nabla_\nu\, T^\nu_\mu &= \partial_\nu\left(\sqrt{g}\,T^\nu_\mu\right) - \frac{1}{2}\,\partial_\mu\, g_{\alpha\beta}\sqrt{g}\,T^{\alpha\beta} \\ &= \partial_\nu\left(\sqrt{g}\,T^\nu_\mu + t^\nu_\mu\right) \end{aligned}}$$

Poisson eq + Maxwell eqs suggests $\boxed{G_{\mu\nu}(g) = \kappa T_{\mu\nu}}$

Tensor calculus, Riemannian geometry, Ricci tensor (with Marcel Grossmann)

Einstein’s Zurich Notebook
Vol 1 The Genesis of General Relativity edited by J. Renn



$$\begin{bmatrix}\mu\nu\\ l\end{bmatrix} = \frac{1}{2}\left(\frac{\partial g_{\mu l}}{\partial x_\nu} + \frac{\partial g_{\nu l}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_l}\right) \qquad \frac{\partial}{\partial x_\kappa}\begin{bmatrix} i\,\text{[illegible]}\\ \text{[illegible]}\end{bmatrix} - \frac{\partial}{\partial x_i}\begin{bmatrix}\kappa\,\text{[illegible]}\\ \text{[illegible]}\end{bmatrix}$$

$$(i\kappa, lm) = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x_\kappa \partial x_l} + \frac{\partial^2 g_{\kappa l}}{\partial x_i \partial x_m} - \frac{\partial^2 g_{il}}{\partial x_\kappa \partial x_m} - \frac{\partial^2 g_{\kappa m}}{\partial x_i \partial x_l}\right) + \sum_{\rho\sigma}\gamma_{\rho\sigma}\left(\begin{bmatrix}im\\ \sigma\end{bmatrix}\begin{bmatrix}\kappa l\\ \rho\end{bmatrix} - \begin{bmatrix}il\\ \sigma\end{bmatrix}\begin{bmatrix}\kappa m\\ \rho\end{bmatrix}\right)$$

Grossmann
Tensor vierter
Mannigfaltigkeit

$$\sum \gamma_{\kappa l}(i\kappa, lm) \quad ?$$

$$\sum \gamma_{\kappa l}\begin{bmatrix}\kappa l\\ \rho\end{bmatrix} = \sum \gamma_{\kappa l}\left[\frac{\partial g_{\kappa\rho}}{\partial x_l} + \frac{\partial g_{l\rho}}{\partial x_\kappa} - \frac{\partial g_{\kappa l}}{\partial x_\rho}\right]$$

$$= \cancel{\tfrac{1}{2}}\,\frac{\partial \lg G}{\partial x_\rho} + 2\sum_{\kappa l}\gamma_{\kappa l}\frac{\partial g_{\kappa\rho}}{\partial x_l}$$

$$\frac{1}{4}\sum \gamma_{\rho\sigma}\left(\frac{\partial g_{i\sigma}}{\partial x_m} + \frac{\partial g_{m\sigma}}{\partial x_i} - \frac{\partial g_{im}}{\partial x_\sigma}\right)\left[-\frac{\partial \lg G}{\partial x_\rho} + 2\sum_{\kappa l}\gamma_{\kappa l}\frac{\partial g_{\kappa\rho}}{\partial x_l}\right]$$

$$\sum \gamma_{\kappa l}\gamma_{\rho\sigma}\left(\begin{bmatrix}im\\ \rho\end{bmatrix}\begin{bmatrix}\kappa l\\ \sigma\end{bmatrix} - \begin{bmatrix}il\\ \sigma\end{bmatrix}\begin{bmatrix}\kappa m\\ \rho\end{bmatrix}\right)$$

$$= \sum_\rho -\begin{Bmatrix}im\\ \rho\end{Bmatrix}\cdot\frac{\partial \lg G}{\partial x_\rho} + 2\sum_{\kappa l\rho}\begin{Bmatrix}im\\ \rho\end{Bmatrix}\cdot\gamma_{\kappa l}\frac{\partial g_{\kappa\rho}}{\partial x_l} - \sum_{\rho l\kappa}\begin{Bmatrix}il\\ \rho\end{Bmatrix}\left(\frac{\partial g_{\kappa\rho}}{\partial x_m}\right)\gamma_{\kappa l}$$

$$+ \sum_{\rho l}\begin{Bmatrix}il\\ \rho\end{Bmatrix}\cdot\begin{Bmatrix}\rho m\\ l\end{Bmatrix}$$

$$\sum_\kappa\left(\frac{\partial^2 g_{\kappa\kappa}}{\partial x_i \partial x_m} - \frac{\partial^2 g_{i\kappa}}{\partial x_\kappa \partial x_m} - \frac{\partial^2 g_{m\kappa}}{\partial x_\kappa \partial x_i}\right) = 0$$

Sollte verschwinden.

$$\varphi = \sum_{im\kappa l}\gamma_{im}\gamma_{\kappa l}\left(\frac{\partial^2 g_{im}}{\partial x_\kappa \partial x_l} - \frac{\partial^2 g_{il}}{\partial x_\kappa \partial x_m}\right)$$

$$+ \sum_{\rho\sigma im\kappa l}\gamma_{\rho\sigma}\gamma_{im}\gamma_{\kappa l}\left(\begin{bmatrix}im\\ \sigma\end{bmatrix}\begin{bmatrix}\kappa l\\ \rho\end{bmatrix} - \begin{bmatrix}il\\ \sigma\end{bmatrix}\begin{bmatrix}\kappa m\\ \rho\end{bmatrix}\right)$$

$$\sum_{\rho\sigma im\kappa l}\gamma_{\rho\sigma}\gamma_{im}\gamma_{\kappa l}\left(\frac{\partial g_{i\sigma}}{\partial x_m} + \frac{\partial g_{m\sigma}}{\partial x_i} - \frac{\partial g_{im}}{\partial x_\sigma}\right)\left(\frac{\partial g_{\kappa\rho}}{\partial x_l} + \frac{\partial g_{l\rho}}{\partial x_\kappa} - \frac{\partial g_{\kappa l}}{\partial x_\rho}\right)$$

$$-g_{i\sigma}\frac{\partial \gamma_{im}}{\partial x_m} \quad -g_{m\sigma}\frac{\partial \gamma_{im}}{\partial x_i} \quad -\gamma_{im}\frac{\partial g_{im}}{\partial x_\sigma} \quad -g_{\kappa\rho}\frac{\partial \gamma_{\kappa l}}{\partial x_l} \quad -g_{l\rho}\frac{\partial \gamma_{\kappa l}}{\partial x_\kappa} \quad -\gamma_{\kappa l}\frac{\partial g_{\kappa l}}{\partial x_\rho}$$

$$\sum \gamma_{\rho\sigma}\Bigg(\underbrace{g_{i\sigma}\frac{\partial \gamma_{im}}{\partial x_m} + g_{m\sigma}\frac{\partial \gamma_{im}}{\partial x_i}}_{\underbrace{\frac{\partial \gamma_{\sigma m}}{\partial x_m} + \frac{\partial \gamma_{i\sigma}}{\partial x_i}}_{2\frac{\partial \gamma_{\sigma\alpha}}{\partial x_\alpha}}} + \underbrace{\gamma_{im}\frac{\partial g_{im}}{\partial x_\sigma}}_{\gamma_{\rho\sigma}\frac{\partial \lg G}{\partial x_\sigma}}\Bigg)\Bigg(g_{\kappa\rho}\frac{\partial \gamma_{\kappa l}}{\partial x_l} + g_{l\rho}\frac{\partial \gamma_{\kappa l}}{\partial x_\kappa} + \underbrace{\gamma_{\kappa l}\frac{\partial g_{\kappa l}}{\partial x_\rho}}_{\frac{\partial \lg G}{\partial x_\rho}}\Bigg)$$

$$\underline{\gamma_{\rho\sigma}\frac{\partial \lg G}{\partial \sigma}\frac{\partial \lg G}{\partial \rho}} + \gamma_{\rho\sigma}\frac{\partial \lg G}{\partial x_\sigma}\left(g_{\kappa\rho}\frac{\partial \gamma_{\kappa l}}{\partial x_l} + g_{l\rho}\frac{\partial \gamma_{\kappa l}}{\partial x_\kappa}\right) + \frac{\partial \lg G}{\partial x_\rho}\cdot 2\cdot\frac{\partial \gamma_{\rho\alpha}}{\partial x_\alpha}$$

$$\frac{\partial \lg G}{\partial x_\sigma}\left(\frac{\partial \gamma_{\rho\alpha}}{\partial x_\alpha} + \frac{\partial \gamma_{\sigma\alpha}}{\partial x_\alpha}\right)$$

$$\underline{3\frac{\partial \lg G}{\partial x_\sigma}\frac{\partial \gamma_{\sigma\alpha}}{\partial x_\alpha} + \frac{\partial \lg G}{\partial x_\sigma}\frac{\partial \gamma_{\rho\alpha}}{\partial x_\alpha}}$$

$$+ 2\frac{\partial \gamma_{\rho\alpha}}{\partial x_\alpha}\left(g_{\kappa\rho}\frac{\partial \gamma_{\kappa l}}{\partial x_l} + g_{l\rho}\frac{\partial \gamma_{\kappa l}}{\partial x_\kappa}\right)$$

$$\cancel{4\frac{\partial \gamma_{\rho\alpha}}{}}\;\cancel{4 g_{\kappa\rho}\frac{\partial \gamma_{\kappa\alpha}}{\partial x_\alpha}\frac{\partial \gamma_{\rho\beta}}{\partial x_\beta}}$$

$l\rho \quad \kappa\alpha$
$\kappa\rho \quad \alpha\beta$

$$4 g_{\kappa\rho}\frac{\partial \gamma_{\kappa\alpha}}{\partial x_\alpha}\frac{\partial \gamma_{\rho\beta}}{\partial x_\beta} + \cancel{2 g_{\kappa\rho}\frac{\partial \gamma_{\kappa\alpha}}{\partial x_\alpha}} \quad \Big/ \cdot\frac{1}{4}$$

war richtig.

Obstacles towards building GR

EP → «Rindler» $ds^2=-c^2(x)dt^2+dx^2+dy^2+dz^2$: strongly suggests that the newtonian limit involves

$$\boxed{g_{00} \simeq -\left(1-\frac{U}{c^2}\right)^2 \quad , \quad g_{ij} \simeq \delta_{ij}}$$

Mach’s principle suggests that $T_{\mu\nu}$ (matter) uniquely determines $g_{\mu\nu}$ (inertia)

Conflict with general covariance (« hole argument »)

Looks for an intermediate (g-dependent) EP-GR covariance groupLinearized version of Ricci tensor incompatible with expected Newtonian limit.

Already in 1912-1913 stumbles on $R^L_{\mu\nu} - \frac{1}{2}R^L\,\eta_{\mu\nu} \sim \Box\left(h_{\mu\nu} - \frac{1}{2}h\,\eta_{\mu\nu}\right) \sim T_{\mu\nu}$

Problem of the meaning of coordinates (especially after SR’s breakthrough)

Unknown contracted Bianchi identitires

Unknown Noether-type theorems

Unknown concept of connection $\Gamma^{\lambda}_{\mu\nu}$

No definition of manifold (Weyl 1913 Concept of Riemann surface)

Unfashionable action principles (even Pauli in 1921)

No good notation, no summation convention

1913 Einstein’s – Grossmann « outline theory »

Semingly impossible to satisfy all the wished physical and mathematical requirements with generally covevariant field equations.

Exclude field equations based on $G_{\mu\nu} \sim R_{\mu\nu}$

Looks for theory with restricted covariance but energy conservation and **strong** EP

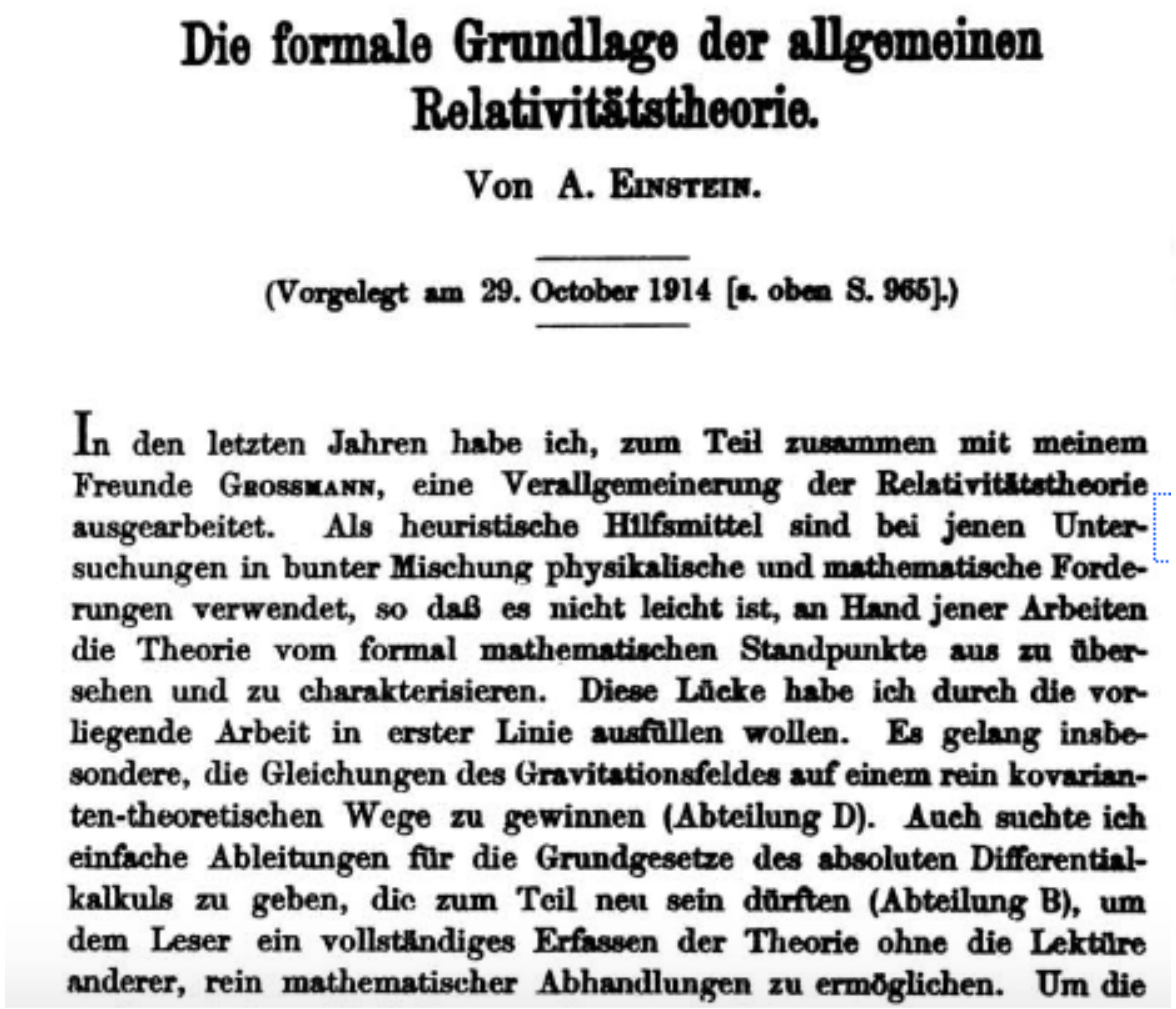
$$\sum_{\alpha\beta\mu} \frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\,\gamma_{\alpha\beta}\,g_{\sigma\mu}\,\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right) = \varkappa(\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu});$$

with $-2\varkappa \cdot \mathfrak{t}_{\sigma\nu} = \sqrt{-g}\left(\sum_{\beta\tau\rho} \gamma_{\beta\nu}\,\frac{\partial g_{\tau\rho}}{\partial x_\sigma}\,\frac{\partial \gamma_{\tau\rho}}{\partial x_\beta} - \frac{1}{2}\sum_{\alpha\beta\tau\rho} \delta_{\sigma\nu}\,\gamma_{\alpha\beta}\,\frac{\partial g_{\tau\rho}}{\partial x_\alpha}\,\frac{\partial \gamma_{\tau\rho}}{\partial x_\beta}\right);$

With $\sum_\nu \frac{\partial}{\partial x_\nu}(\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu}) = 0.$

? What is the restricted covariance group ?

Einstein’s October 1914 paper

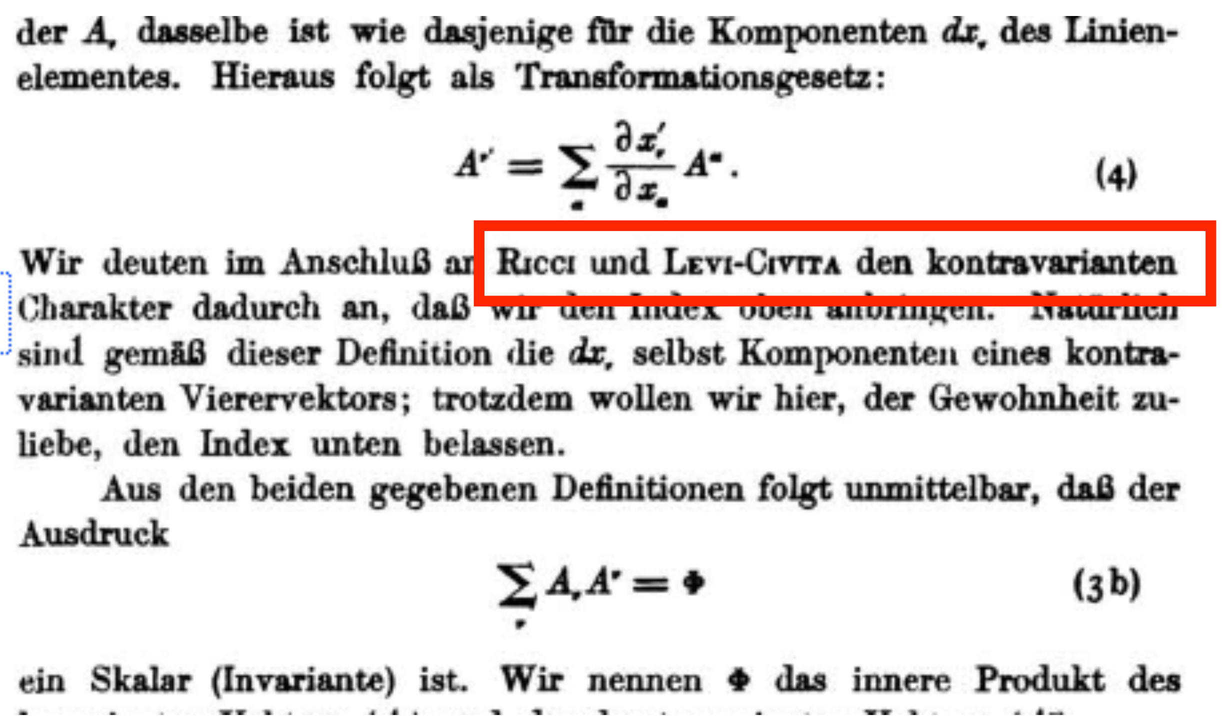


Die formale Grundlage der allgemeinen Relativitätstheorie.

Von A. Einstein.

(Vorgelegt am 29. October 1914 [s. oben S. 965].)

In den letzten Jahren habe ich, zum Teil zusammen mit meinem Freunde Grossmann, eine Verallgemeinerung der Relativitätstheorie ausgearbeitet. Als heuristische Hilfsmittel sind bei jenen Untersuchungen in bunter Mischung physikalische und mathematische Forderungen verwendet, so daß es nicht leicht ist, an Hand jener Arbeiten die Theorie vom formal mathematischen Standpunkte aus zu übersehen und zu charakterisieren. Diese Lücke habe ich durch die vorliegende Arbeit in erster Linie ausfüllen wollen. Es gelang insbesondere, die Gleichungen des Gravitationsfeldes auf einem rein kovarianten-theoretischen Wege zu gewinnen (Abteilung D). Auch suchte ich einfache Ableitungen für die Grundgesetze des absoluten Differentialkalkuls zu geben, die zum Teil neu sein dürften (Abteilung B), um dem Leser ein vollständiges Erfassen der Theorie ohne die Lektüre anderer, rein mathematischer Abhandlungen zu ermöglichen. Um die



der A_ν dasselbe ist wie dasjenige für die Komponenten dx_ν des Linienelementes. Hieraus folgt als Transformationsgesetz:

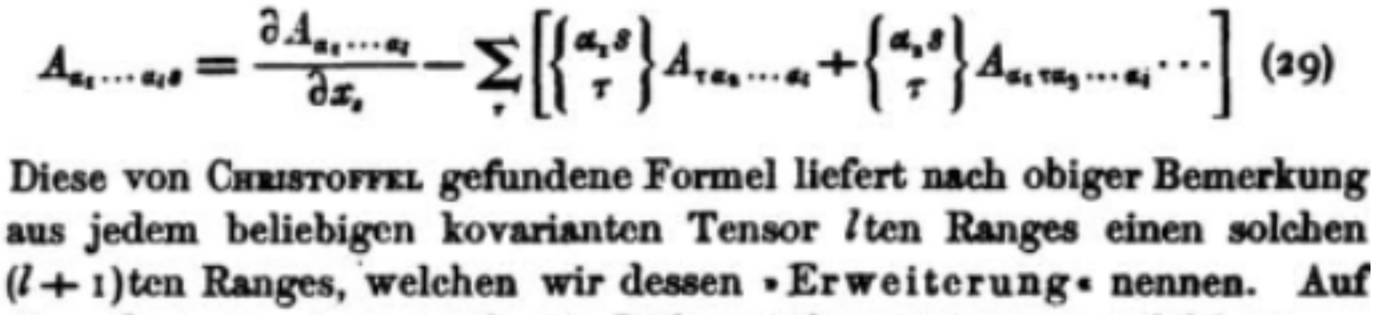
$$A^{\sigma\prime} = \sum_\nu \frac{\partial x'_\sigma}{\partial x_\nu} A^\nu. \qquad (4)$$

Wir deuten im Anschluß an Ricci und Levi-Civita den kontravarianten Charakter dadurch an, daß wir den Index oben anbringen. Natürlich sind gemäß dieser Definition die dx_ν selbst Komponenten eines kontravarianten Vierervektors; trotzdem wollen wir hier, der Gewohnheit zuliebe, den Index unten belassen.

Aus den beiden gegebenen Definitionen folgt unmittelbar, daß der Ausdruck

$$\sum_\nu A_\nu A^\nu = \Phi \qquad (3b)$$

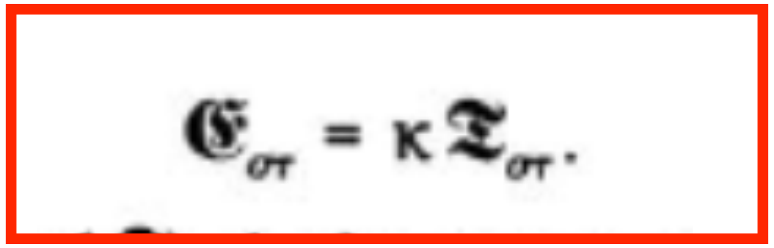
ein Skalar (Invariante) ist. Wir nennen Φ das innere Produkt des



$$A_{\alpha_1\cdots\alpha_l s} = \frac{\partial A_{\alpha_1\cdots\alpha_l}}{\partial x_s} - \sum_\tau \left[\left\{\begin{matrix}\alpha_1 s\\ \tau\end{matrix}\right\} A_{\tau\alpha_2\cdots\alpha_l} + \left\{\begin{matrix}\alpha_2 s\\ \tau\end{matrix}\right\} A_{\alpha_1\tau\alpha_3\cdots\alpha_l}\cdots\right] \qquad (29)$$

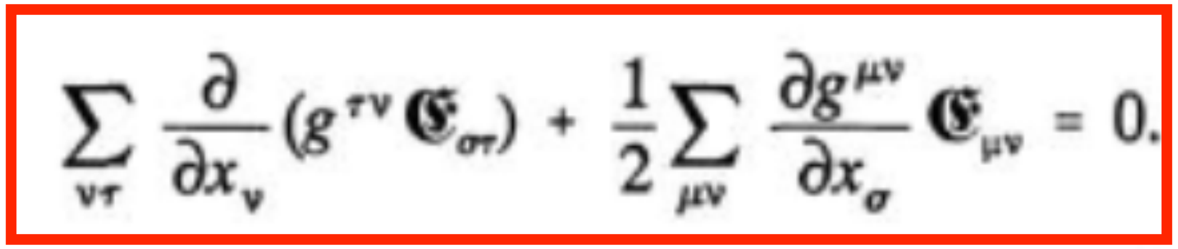
Diese von Christoffel gefundene Formel liefert nach obiger Bemerkung aus jedem beliebigen kovarianten Tensor $l$ten Ranges einen solchen $(l+1)$ten Ranges, welchen wir dessen »Erweiterung« nennen. Auf

Einstein looks for field eqs of the type



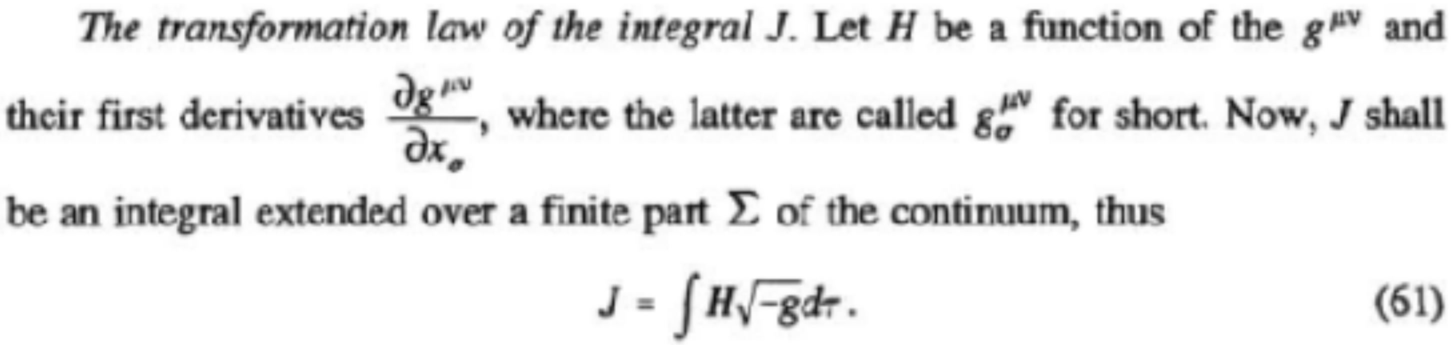
$$\mathfrak{G}_{\sigma\tau} = \kappa\,\mathfrak{T}_{\sigma\tau}.$$

with the lhs satisfying



$$\sum_{\nu\tau} \frac{\partial}{\partial x_\nu}(g^{\tau\nu}\mathfrak{G}_{\sigma\tau}) + \frac{1}{2}\sum_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial x_\sigma}\mathfrak{G}_{\mu\nu} = 0.$$

deriving from an action principle



The transformation law of the integral J. Let H be a function of the $g^{\mu\nu}$ and their first derivatives $\frac{\partial g^{\mu\nu}}{\partial x_\sigma}$, where the latter are called $g^{\mu\nu}_\sigma$ for short. Now, J shall be an integral extended over a finite part Σ of the continuum, thus

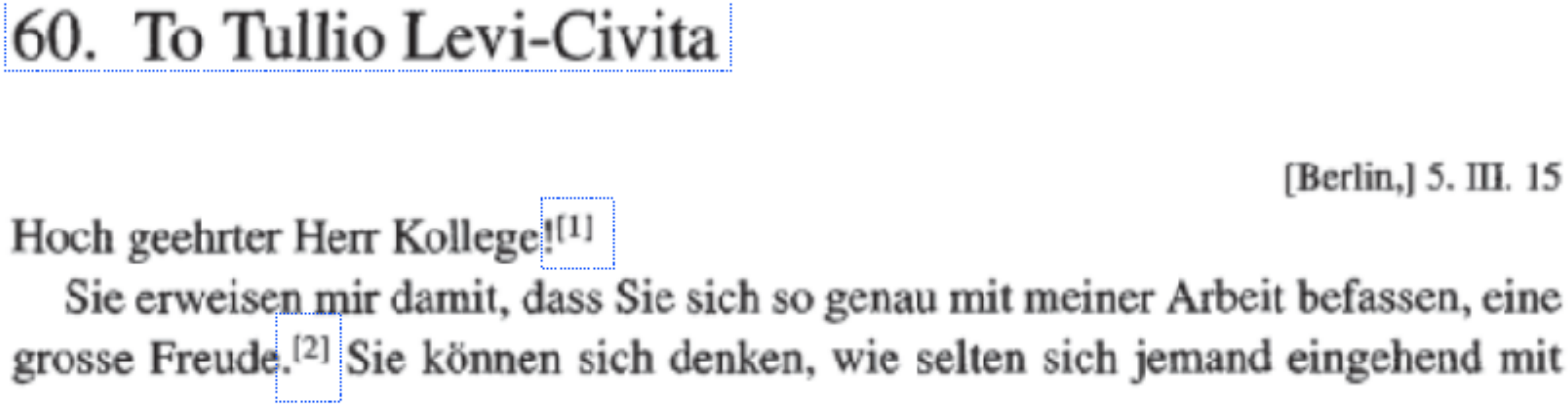
$$J = \int H\sqrt{-g}\,d\tau. \qquad (61)$$

with an invariance under a group of « adapted coordinates », larger than linear transformations but smaller than all diffeomorphisms

The Einstein-Levi-Civita Correspondence in March-May 1915

In February 1915 Max Abraham attracts the attention of LC to Einstein's work on generalizing relativity to more general changes of frames and in particular on Einstein's October 1914 paper (Judith Goodstein, 1983)

This prompts LC to write to Einstein in March 1915; Einstein answers very fast:



60. To Tullio Levi-Civita

[Berlin,] 5. III. 15

Hoch geehrter Herr Kollege![1]

Sie erweisen mir damit, dass Sie sich so genau mit meiner Arbeit befassen, eine grosse Freude.[2] Sie können sich denken, wie selten sich jemand eingehend mit

There ensues an exchange of ~ 2x11 letters between E and LC:
though LC is criticizing (and putting in doubt) the main claim of E,
this very technical correspondence is quite friendly and open-minded.
E is very happy to have attracted the attention of « an esteemed collegue » ,
but stubbornly defends his reasonings and claims.

In his second letter E writes: « I would be very glad if you would write to me next time in Italian. As a young man I spent over half a year in Italy and at that time I also had the pleasure to visit the charming little town of Padua, and I still enjoy making use of the Italian language. »

In a PS to his 2 April 1915 letter to LC, E adds: « I never had a correspondence as interesting as this before. You should see how I always look forward to your letters. »

Useful interaction and missed opportunities

In spite of E’s stubborness (see G. Weinstein 2012), this correspondence was useful for both correspondents:
E rethought the basic assumptions of his search for a generalized theory of relativity.
LC had been prepared to be one of the first important contributors ot GR

Without entering into the details of the technical discussion (which were actually quite subtle), let us note two missed opportunities during this discussion:

1. LC never noticed a basic technical error in E’s calculations, namely (Hilbert Nov. 1915):

$$\frac{\partial}{\partial x^\mu}\Delta g^{\alpha\beta} \equiv \frac{\partial}{\partial x^\mu}\left(g'^{\alpha\beta}(x+\xi(x)) - g^{\alpha\beta}(x)\right)$$

$$\neq \Delta\frac{\partial g^{\alpha\beta}}{\partial x^\mu} = \frac{\partial g'^{\alpha\beta}}{\partial x'^\mu}(x+\xi(x)) - \frac{\partial g^{\alpha\beta}}{\partial x^\mu}(x)$$

2. LC did not mention (did not know ?) that the basic requirement of E for the gravitational tensor on the lhs: $G_{\mu\nu} = \kappa T_{\mu\nu}$

$$\boxed{\nabla^\nu G_{\mu\nu} = 0}$$

was satisfied by
(contracted Bianchi ids)

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}$$

3. LC did not mention (did not realize?, like Hilbert) to E the inconclusiveness of his (« hole ») causality argument: invariance under diffeomorphims is not a lack of physical causality.

The 1915 breakthrough

June –July 1915 Hilbert invites Einstein to give, during one week, six 2-hour lectures on General Relativity.

Hilbert gets henceforth interested in incorporating Einstein's ideas within his attempt at an all-encompassing axiomatic approach to physics (à la Gustav Mie).

4 November 1915 Restricts full covariance to unimodular diffeomorphisms Jacobian=1
Decomposes $R_{\mu\nu} = R^{\mathrm{I}}_{\mu\nu} + R^{\mathrm{II}}_{\mu\nu} \quad , \quad R^{\mathrm{I}}_{\mu\nu} = \partial_\sigma \Gamma^\sigma_{\mu\nu} - \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\rho\nu}$

and shows that $R^{\mathrm{I}}_{\mu\nu} = \kappa T_{\mu\nu}$ derives from an action and admits a conservation law.

11 November 1915 shows that, under the assumption T = 0, one can write the fully covariant $R_{\mu\nu} = \kappa T_{\mu\nu}$ (which can be simplified to the 4 November form by imposing $\sqrt{g} = 1$)

18 November 1915 solves $R_{\mu\nu} = 0 \ , \sqrt{g} = 1$ outside the Sun to second order in $1/c^2$ and finds an advance of Mercury' perihelion equal to 43''/cy and an unexpected Newtonian limit with $g_{ij} \neq \delta_{ij}$.

The 1915 breakthrough (2)

25 November 1915 He writes the general (T ≠ 0), final generally covariant equations

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)$$

and proves (when simplifying the calculation by using unimodular coordinates √g=1) that they imply the energy conservation

$$\sum_{\lambda} \frac{\partial}{\partial x_{\lambda}} \left(T_{\sigma}^{\lambda} + t_{\sigma}^{\lambda} \right) = 0$$

So that the strong EP is satisfied : the source of gravity is $T_{\mu}^{\nu} + t_{\mu}^{\nu}$

NB: Einstein thereby gave a direct computational proof of the « contracted Bianchi identity »

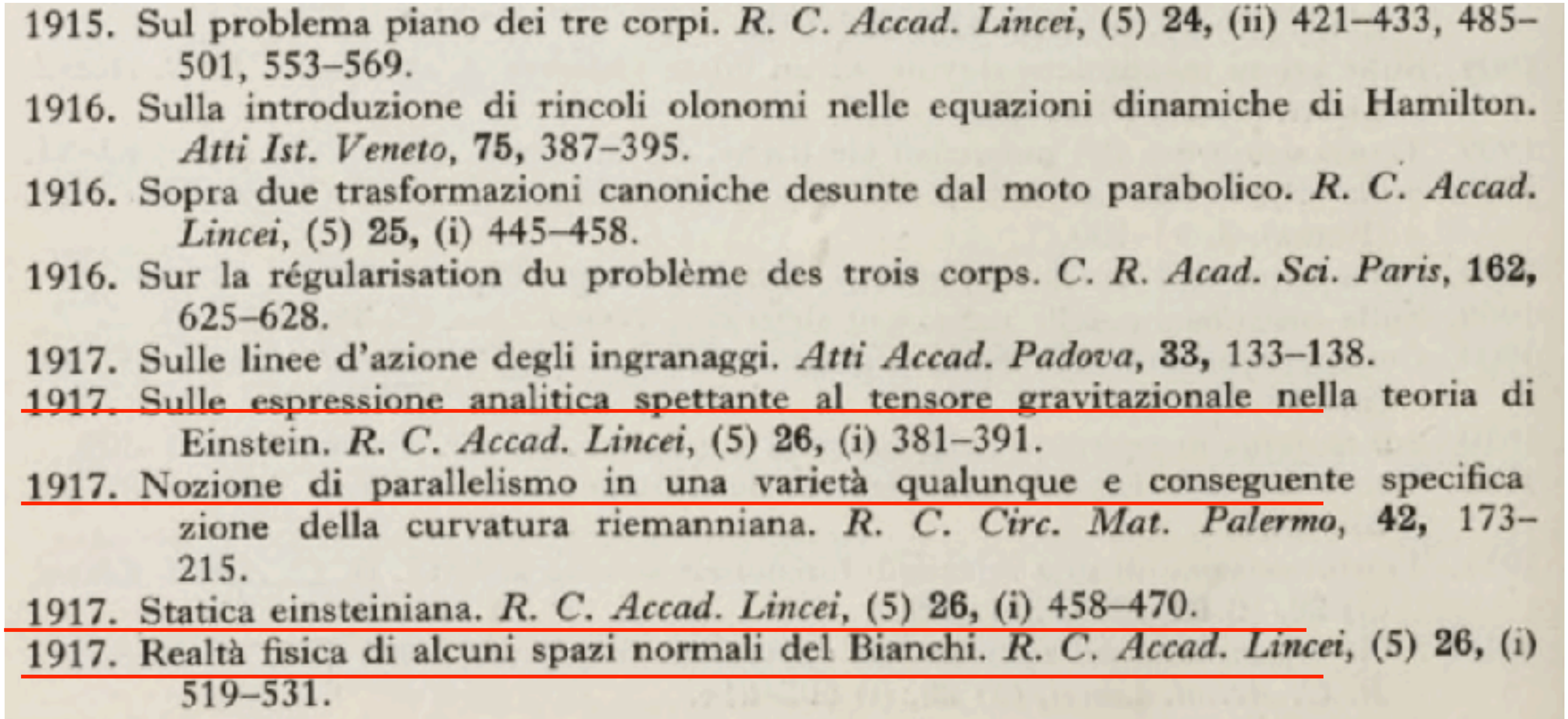
$$\boxed{\nabla^{\nu}(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}) \equiv 0}$$

The Hilbert affair

On 20 November 1915 (after the first three communications of Einstein and before the final one of 25 November) Hilbert presents a communication to the Göttingen Academy. The published version of his communication includes both the Einstein-Hilbert action for gravity $\int d^4x\,\sqrt{g}\,R$ and the explicit form of Einstein's field equations $R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = \kappa T_{\mu\nu}$. For a long time it was concluded that Hilbert had partially « scooped » Einstein in getting first the final field equations of GR. However, Corry, Renn, and Stachel (1997) found the first proofs of Hilbert's paper (December 1915).

These proofs (despite the theft of the a fraction of one page !) show that :

1. Hilbert has substantially amended / completed his paper between submission and publication
2. Hilbert initially postulated the necessity of breaking general covariance
3. Hilbert correctly postulated the action $\int d^4x\,\sqrt{g}\,R$ but very probably did not obtain the explicit form of Einstein's field equations until he saw Einstein's last paper.



1915. Sul problema piano dei tre corpi. *R. C. Accad. Lincei*, (5) **24**, (ii) 421–433, 485–501, 553–569.

1916. Sulla introduzione di vincoli olonomi nelle equazioni dinamiche di Hamilton. *Atti Ist. Veneto*, **75**, 387–395.

1916. Sopra due trasformazioni canoniche desunte dal moto parabolico. *R. C. Accad. Lincei*, (5) **25**, (i) 445–458.

1916. Sur la régularisation du problème des trois corps. *C. R. Acad. Sci. Paris*, **162**, 625–628.

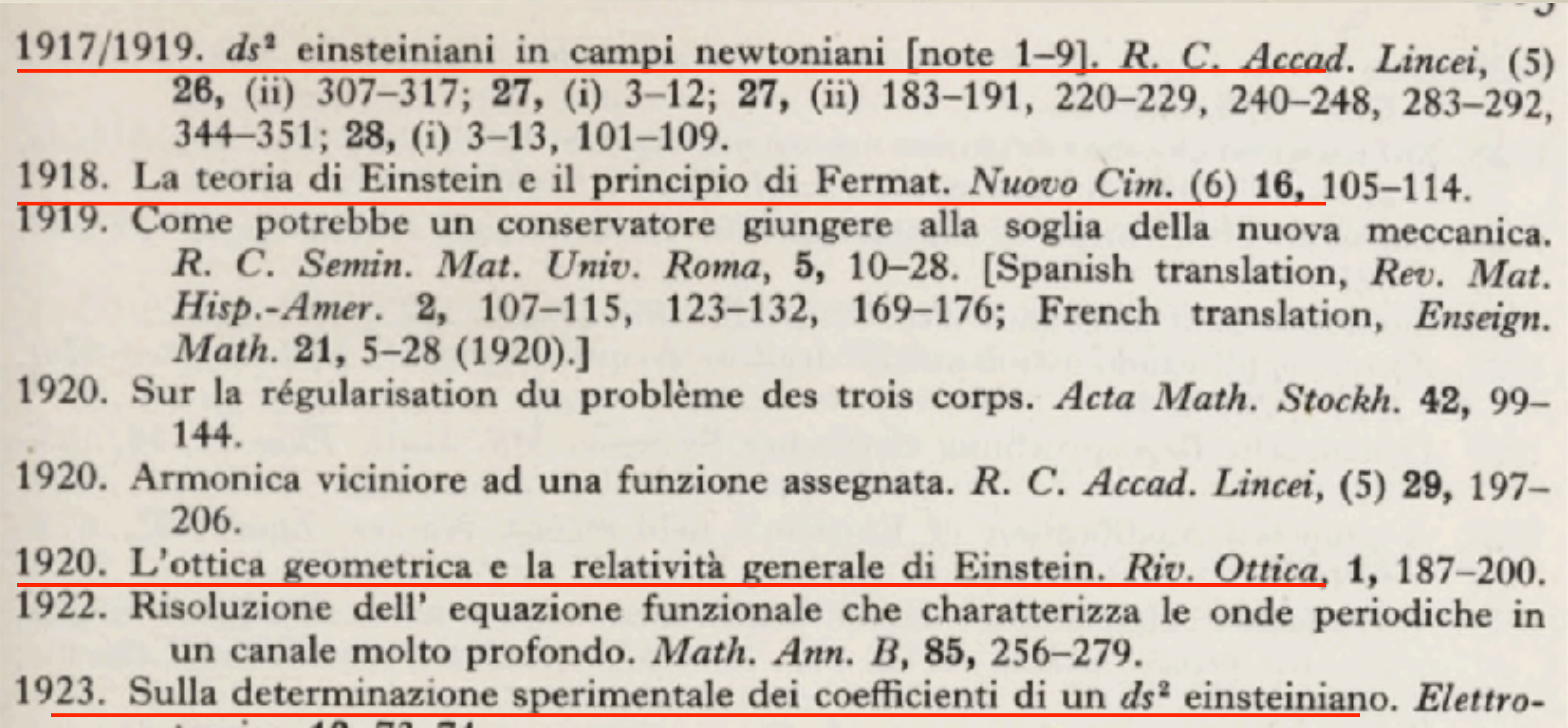
1917. Sulle linee d'azione degli ingranaggi. *Atti Accad. Padova*, **33**, 133–138.

1917. Sulle espressione analitica spettante al tensore gravitazionale nella teoria di Einstein. *R. C. Accad. Lincei*, (5) **26**, (i) 381–391.

1917. Nozione di parallelismo in una varietà qualunque e conseguente specificazione della curvatura riemanniana. *R. C. Circ. Mat. Palermo*, **42**, 173–215.

1917. Statica einsteiniana. *R. C. Accad. Lincei*, (5) **26**, (i) 458–470.

1917. Realtà fisica di alcuni spazi normali del Bianchi. *R. C. Accad. Lincei*, (5) **26**, (i) 519–531.



1917/1919. ds^2 einsteiniani in campi newtoniani [note 1–9]. *R. C. Accad. Lincei*, (5) **26**, (ii) 307–317; **27**, (i) 3–12; **27**, (ii) 183–191, 220–229, 240–248, 283–292, 344–351; **28**, (i) 3–13, 101–109.

1918. La teoria di Einstein e il principio di Fermat. *Nuovo Cim.* (6) **16**, 105–114.

1919. Come potrebbe un conservatore giungere alla soglia della nuova meccanica. *R. C. Semin. Mat. Univ. Roma*, **5**, 10–28. [Spanish translation, *Rev. Mat. Hisp.-Amer.* **2**, 107–115, 123–132, 169–176; French translation, *Enseign. Math.* **21**, 5–28 (1920).]

1920. Sur la régularisation du problème des trois corps. *Acta Math. Stockh.* **42**, 99–144.

1920. Armonica viciniore ad una funzione assegnata. *R. C. Accad. Lincei*, (5) **29**, 197–206.

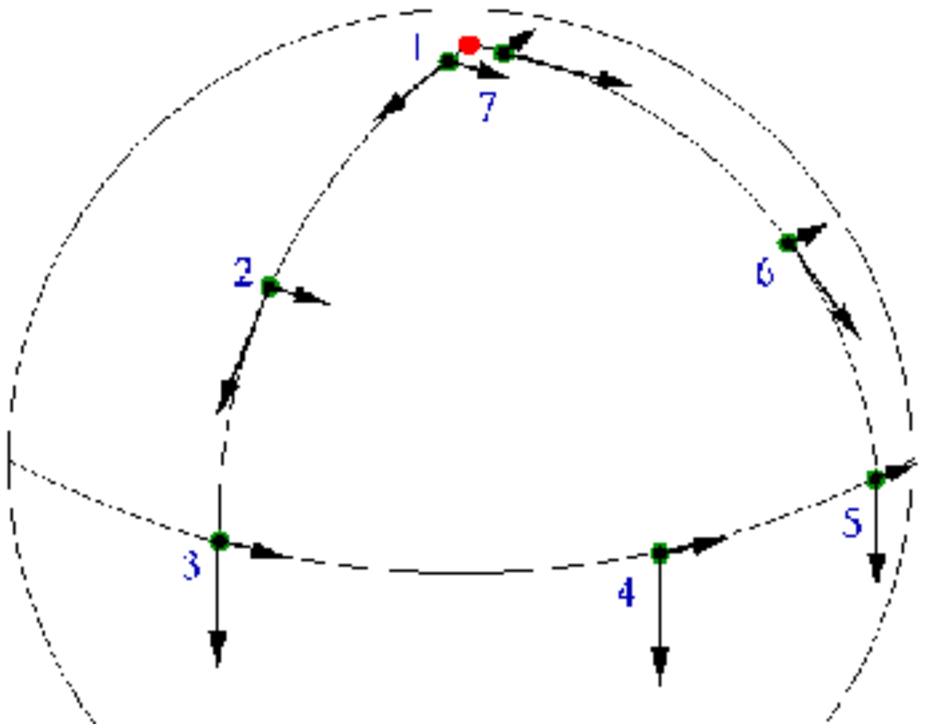
1920. L'ottica geometrica e la relatività generale di Einstein. *Riv. Ottica*, **1**, 187–200.

1922. Risoluzione dell' equazione funzionale che charatterizza le onde periodiche in un canale molto profondo. *Math. Ann. B*, **85**, 256–279.

1923. Sulla determinazione sperimentale dei coefficienti di un ds^2 einsteiniano. *Elettro-*

The notion of parallel transport (1917)

$$\begin{tikzcd}
\text{metric}\ g_{\mu\nu} \arrow[rd] \arrow[dd, "\{^{\lambda}{}_{\mu\nu}\}"', "\nabla_{\mu}"] & \\
& \Gamma^{\lambda}_{\mu\nu}\ \text{connection} \arrow[ld] \\
\text{curvature}\ R_{\mu\nu\rho\sigma} &
\end{tikzcd}$$



Gave a deeper understanding of curvature (as non-integrability of parallel transport)

Opened the way to important generalizations:
affine connections, general connections, ….

Developed by many people, notably Weyl, Schouten, Struik, Cartan, Einstein, Ehresmann, Yang-Mills, ……

Statica Einsteiniana (1917)

$$\boxed{ds^2 = -V^2(x)dt^2 + g_{ij}(x)dx^i dx^j}$$

LC derives the static form of Einstein’s field equations:

$$R^{(3)} = 2\kappa\rho$$

$$R^{(3)}_{ij} - \frac{1}{2}R^{(3)}g_{ij} - \frac{\nabla_{ij}V}{V} + \frac{\Delta V}{V}g_{ij} = \kappa T_{ij}$$

[NB: LC calls R^(3)_ij-1/2R^(3)g_ij the Ricci tensor, and R^(n)_ij the Einstein tensor!]

He then finds explicit solutions with « Bianchi-normal » three- metrics, notably
the « Bertotti-Robinson (1959) » metric (uniform electric field),
and the cylindrical LC metric (spatial) « Kasner (1921) »

$$\boxed{\mathrm{d}s^2 = -r^{-2h}\left[r^{2h^2}\left(\mathrm{d}r^2 + \mathrm{d}z^2\right) + \frac{r^2}{C^2}\,\mathrm{d}\varphi^2\right] + r^{2h}\mathrm{d}t^2,}$$

Einstein’s letter to LC (Luzern 2 August 1917):

« I admire the elegance of your method of calculation. It must be nice to ride through these fields upon the horse of true mathematics, while the like of us have to make our way laboriously on foot. »

T. Levi-Civita on Gravitational Waves

Caratteristiche e bicaratteristiche delle equazioni gravitazionali di Einstein I, II, Rend. Accad Naz. Lincei (Sci. Fis. Mat. Nat.) 1(1930) 3, 11

This is an early understanding of the exact propagation properties of Einsteinian gravity (i.e gravitational waves), after Darmois (1927) but before Stellmacher (1938) and Lichnerowicz (1939).
A striking fact is that mathematicians had clarified the mathematical existence of gravitational waves, while it took many more years for physicists to convince themselves of their reality (cf Bondi, Pirani, Infeld, … in the 1950's)

Levi-Civita on Dirac’s equation in curved spacetime

Diracsche und Schrödingersche Gleichungen
Aus Sitzungsberichte d. Preuss. Akad. d.
Wiss. Phys.-math. Kl. 1933, 5

According to Hodge’s obituary notice, LC proposed to replace Dirac’s first order equations by a set of second-order equations which took into account the gravitational field. However, LC’s equations are only reconcilable with Dirac’s (in absence of gravity) for special electromagnetic fields (purely electric or magnetic).

A missed opportunity to apprehend the role of the Ricci rotation coefficients (i.e the Levi-Civita connection referred to a moving frame, or vierbien) in defining the parallel transport (o=and covariant derivative) of a spinor.

The long-forgotten Ricci rotation coefficients !

In: Méthodes de calcul différentiel absolu et leurs applications
Ricci, M.M.G.; T., Levi-Civita (Mathematische Annalen 54 , 1901)
an important role is played by the use of a field of orthonormal frames (vierbein, repere mobile, moving frame) and by the corresponding value of the (Levi-Civita) connection: the Ricci rotation coefficients:

$$\gamma_{hk;l} = \nabla_\nu e_{h\mu}\, e_k^\mu e_l^\nu$$

$$\gamma_{hk;j} = -\gamma_{kh;j}$$

$$\nabla_{e_j} e_h^\mu = \gamma_{hi;j}\, e_i^\mu$$

This aspect of the « calcul différentiel absolu » was not needed in the early (second-order) « metric » formulation of GR by Einstein and others. Moreover the 1917 LC definition of parallel transport was also often formulated purely in coordinate frames, i.e with

$$\Gamma^\lambda_{\ \mu\nu} \ \ \text{rather than} \ \ \gamma^i_{\ j;k} \ \ \text{or} \ \ \omega^i_{\ j;\mu}$$

The Ricci rotation coefficients played an important role in geometrical uses of tensor calculus (notably in the hands of E. Cartan), but became important in physics mainly when discussing fermionic fields in curved spaces: from Dirac's equation in curved space, up to modern supergravity theories.

Levi-Civita and the relativistic n-body problem

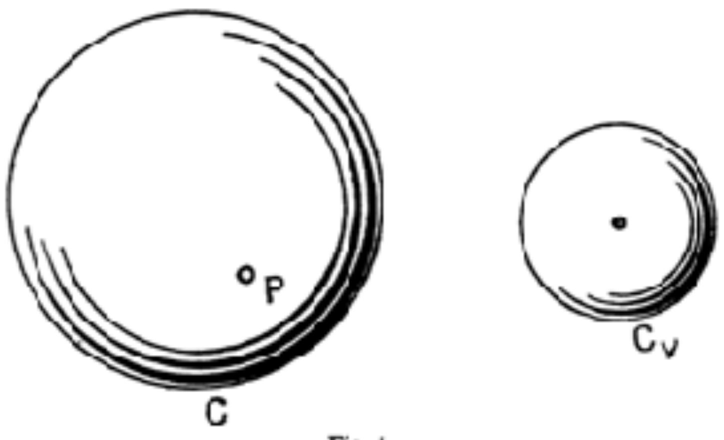


Fig. 1.

On 4 September 1936, at the Harvard Tercentenary Conference, gave preliminary reports on his ongoing work on the relativistic n-body problem.
Arthur Eddington attended LC's talk and was struck by his announced discovery of a putative secular acceleration of the center of mass of a binary system. This spurred Eddington to reexamine the relativistic n-body problem with his student Clark. Eddington-Clark 1938 did not find any secular acceleration, after using a corrected version of the n-body dynamics worked out by De Sitter 1916.
LC also corresponded in the spring of 1938 [see Nastasi-Tazzioli 2005] with Robertson, who was deriving the consequences of the 2-body dynamics derived by Einstein-Infeld-Hoffmann 1938 (submitted 16 June 1937).

LC published his results in two papers [LC 1937a, LC 1937b] and a book: « Le problème des n corps en relativité générale », sent for publication after 1939 [LC cites Fock 1939] but published only in 1950 (Mémorial des sciences mathématiques CXVI)

LC's results « are of a remarkable self-contradictory diversity » (Damour-Schäfer 1988).
A close study of his works shows that LC never succeeded in getting the correct equations of motion for binary systems, neither for the relative, nor for the center of mass motion. [This might be linked to the failing health of LC, and the 1938 political trauma.]
However, one should note several remarkable aspects of LC's work:

1. his conceptually very clear exposition of the n-body problem
2. the pioneering nature of his work
3. his clear understanding of the crucial necessity of proving an « effacement property » (Brillouin) of the eqs of motion of n, extended self-gravitating bodies.

These feats assure him a lasting position among the contributors to relativistic dynamics.

Conclusions

- A huge variety of lasting contributions to different fields
- A productive, very gentlemany, correspondence with Einstein, which led to a friendship.
- Among the ~ 40 articles on General Relativity several of them have turned out to be lasting contributions (not to mention the breakthrough geometrical paper on parallel transport)
- Some missed (probably unavoidable) opportunities.