# The Legacy of Tullio Levi-Civita (three Volumes)

# Padova University Press

This is a publication, in three volumes, devoted to the great mathematician Tullio Levi-Civita. On Friday November 25th, 2016 the Mathematics Department of the University of Padua hosted a dedication ceremony, in the Aula Magna of the Bo Palace, in honour of this eminent scientist. That day three lectures were given and they are collected in the publication.

The first lecture outlining the local historical context in which Levi-Civita began his scientific activity in Padua, his native city, was delivered by Prof. **Carlo Fumian**, Director of the University Research Institute for the History of the Resistance and of the Contemporary Age.

The second lecture was delivered by Prof. **Umberto Bottazzini**, mathematics and science historian from the University of Milan. He described the scientific activity of Levi-Civita during his time in Padua, which covered the first half of his academic life.

In the third and last lecture , Prof. **Claude Viterbo**, Director of the Department of Mathematics and Applications from the École Normale Supérieure de Paris, focused on Levi-Civita's contributions to the three- and N-body problem with applications to the study of planetary movement and on more recent discoveries related to the topic.

## An introduction to the works of Tullio Levi-Civita proposed in this reprint

A reprint of some of Tullio Levi-Civita's work has been supported by the Department of Mathematics and the University of Padua. This initiative pays tribute to a meek gentle Padua native, whose impact was not limited to mathematics but spread also to the physics of gravitation, astronomy and civil engineering.

### Levi-Civita's degree dissertation

The first work we propose consists of an anastatic reprint of the degree dissertation Levi-Civita wrote in 1894, as a conclusion of his student's career:

> Tullio Levi-Civita, *Sugli Invarianti Assoluti* (*On Absolute Invariants*)
>
> Supervisor: Prof. Gregorio Ricci Curbastro

The discovery of the manuscript, in 2016, produced great, unexpected emotion. The precious booklet was hidden in a file in a remote archive of the University of Padua. It comprises about a hundred pages, handwritten by a young Tullio. This thesis proved pivotal to fundamentally renew geometry which, at that time, was called Absolute Differential Calculus. The names of Ricci Curbastro and Levi-Civita are in fact tied universally to this path of renewal. Although Levi-Civita's thesis was immediately published in the Atti del Regio Istituto Veneto di Scienze Lettere ed Arti, we deemed it important and evocative to reprint – in a high quality anastatic copy – the precious historical document.

**Some other papers**

We propose here some short descriptions of Levi-Civita's works collected in the third volume.

> Tullio Levi-Civita, *Interpretazione gruppale degli integrali di un sistema canonico* (*Group-theoretic interpretation of the integrals of a canonical system*). Rend. Acc. Lincei, s. 2^a, col. VIII, 2^0 sem. 1899

The choice of the article is motivated a question which somehow emerged during the preparation (in 2010 – 2012) of a third volume of complements to the reprint of the afore mentioned mechanics manual by Levi-Civita and Amaldi. Grosso modo, the issue can be rephrased as follows: why haven't Levi-Civita and Amaldi ever cited the Theorem by Emmy Noether in their work? This theorem appeared in 1918 both in a finite-dimensional context (analytic mechanics) and in an infinite-dimensional one (field theory). In essence, it states that "A continuous symmetry of the functional in consideration implies a conservation (also called "integral of motion") of some functions of the evolving variables". In this paper, Levi-Civita conceived such formidable connection between "symmetry" and "conservation", although limited to the finite-dimensional Hamiltonian environment, twenty years before the fundamental, far-reaching work by Emmy Noether. By the way, the latter seems to be independent of Levi-Civita's note.

> Gregorio Ricci Curbastro, Tullio Levi-Civita, *Méthodes de calcul différentiel absolu et leurs applications* (*Methods in absolute differential calculus and their applications*) Mathematische Annalen, 54 (1-2), 125-201, 1901.

In 1899 a prominent mathematical pioneer, Felix Klein (Gottingen), asked Ricci Curbastro and Levi-Civita to write an accurate recollection of their Absolute Differential Calculus, to be published in one of the most important mathematical journals to this day, the Mathematische Annalen. It was a noticeable manifestation of esteem by the great German, and in two years the essay *Méthodes de calcul différentiel absolu et leurs applications* was ready for the publisher. The follow-up to the publication of this crucial review paper yielded little cultural satisfaction for Gregorio Ricci Curbastro and Tullio Levi-Civita. Apart from some of the main mathematicians of the time, there was little enthusiasm on the part of international community for the novelty of their absolute differential calculus. However, we will soon see that it did leave a mark on Albert Einstein.

> Tullio Levi-Civita, *Sur la résolution qualitative du problème restreint des trois corps* (*On the qualitative solution of the restricted three-body problem*) Verdhandl. III Intern. Math. Kongresses, Heidelberg, 1904

> Tullio Levi-Civita, *Sur la résolution qualitative du problème restreint des trois corps* (*On the qualitative solution of the restricted three-body problem*)Acta Math., 30 (1906), 305-327.

In the first 15 years of the new century, Tullio Levi-Civita did not work on Absolute Differential Calculus, our modern Differential Geometry. Instead, his attention was mainly captured by Analytical Mechanics, Stability Theory and Celestial Mechanics in particular the Three-Body Problem. The two articles below represent the heart of the so-called "three-body problem

regularization", devised by Levi-Civita, which is still the most efficient regularization available in the literature.

Tullio Levi-Civita, *Nozione di parallelismo in una varietà qualunque e conseguente specificazione geometrica della curvatura Riemanniana (Parallelism in manifolds and geometric specification of the Riemannian curvature)* Rend. Circ. Mat. Palermo (1917)

The year 1915 was the annus mirabilis in which Levi-Civita exchanged letters with Albert Einstein. The theory of absolute differential calculus, developed by Levi-Civita and his mentor Ricci Curbastro, was used by Albert Einstein, as crucial mathematical scaffolding for General Relativity, his revolutionary theory of gravitation. In the first relativity of modern physics, envisioned by Galileo and Newton, the bodies' trajectories are each other deflected because of instantaneous forces generated by their masses, all such objects living in an Euclidean space. Conversely, in Einstein's relativity the trajectories are nothing but geodesics -namely, paths of minimal local length- in a non-Euclidean geometry of the space-time, a geometry which in turn is shaped by the masses. The classical physicist Max Abraham asked Levi-Civita to invalidate the theory of the brilliant German physicist, but, on the contrary, Tullio was absolutely captured by the theory. Yet he found an error in its formulation: the theory was not invariant under local coordinate transformations. A dense correspondence followed between Levi-Civita and Einstein , in which the German physicist strongly defended his equations from the meticulous observations pinpointed by Levi-Civita. A few days before Italy entered the First World War Einstein acknowledged his error. In a letter Albert expresses his heartfelt gratitude to Tullio, and the two men will be longlife connected in a sincere friendship. Soon after this mail exchange Einstein proposed a revolutionary geometrization of gravitational physics: the field generated by the mass distribution is interpreted as a curvature of space-time. Levi-Civita, sensing the universal reach of this approach, identifies its fundamental geometrical side: the notion of *parallel transport* in a curved space. As it turns out, the velocity of a *geodesic* –a curve of minimal length--evolves parallel to itself. Let us mention that parallel transport has proved quite fruitful, for it would eventually become the basic idea for the theory of connections, a crucial branch of modern geometry.

Tullio Levi-Civita, *Come potrebbe un conservatore giungere alla soglia della nuova meccanica (The way for a conservative to reach the threshold of the new mechanics).* Rend, Sem. Mat. Roma (1918/1919)

Starting from the 1st January 1919 Tullio Levi-Civita works at the University of Rome. The paper "Come potrebbe un conservatore giungere alla soglia della nuova meccanica (How can a conservative reach the threshold of new mechanics)" can be considered as a saluting address for the university that is welcoming him. The title and its initial eight lines will touch even the most inexperienced mathematician: Levi-Civita, the progressive, brilliant mathematical innovator advocates for conservatism as a premise for progress, an inescapable feat for whoever strives to advance in research. He accomplishes this through a sort of tale in formulas: showing that the metamorphosis of classical mechanics into general relativity is a necessary process to explain certain experimental confutations, although the classical vision can be recovered as a sharp

approximation of the new theory for velocities much smaller than light. This paper is a gem, for its prowess in divulgation, writing, and ethics of knowledge.

Tullio Levi-Civita, *Détermination rigoureuse des ondes permanentes d'ampleur finie (Rigorous determination of finite-amplitude stationary waves)* Math.Annalen, XCIII (1925), pp. 264-314.

Member of many scientific academies all over the world, Tullio Levi-Civita becomes the mathematician who disseminates General Relativity, by writing articles and essays on it. His research also moves across other fields, like continuum mechanics. Within this discipline, the work *Détermination rigoureuse des ondes permanentes d'ampleur finie* highlights a new line of research.

Tullio Levi-Civita, *Le problème des N-corps en relativité générale (The N-body problem in General-Relativity)* Mémorial des sciences mathématiques; 116 (1950).

Under the fascist racial laws of 1938 Tullio Levi-Civita and many other Jewish researchers are no longer allowed to work in public activities. Levi-Civita was even banned from entering the mathematical library at the University of Rome. It is the beginning of the end: he will die heartbroken in December 1941. During those years, and thanks to the love of his students, he found the strength to write his last recollection on the N-body problem in General Relativity, posthumously published. . This deep, complex work is filled with new ideas and also numerical. Its predictive potential may still not be fully explored.