On the logical structure of some choice, maximality, bar induction, and well-foundedness principles

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Standard reverse mathematics of the axiom of choice in set theory

Three well-known equivalent presentations in set theory:

- **axiom of choice** (AC): any family of non-empty sets has a choice function
- **Zorn's lemma** (ZL): if all chains of a non-empty partially ordered set are bounded upwards, the set has a maximal belement
- **the well-ordering principle**: every set can be well-ordered

and many others:

- e.g. **Teichmüller-Tukey lemma**

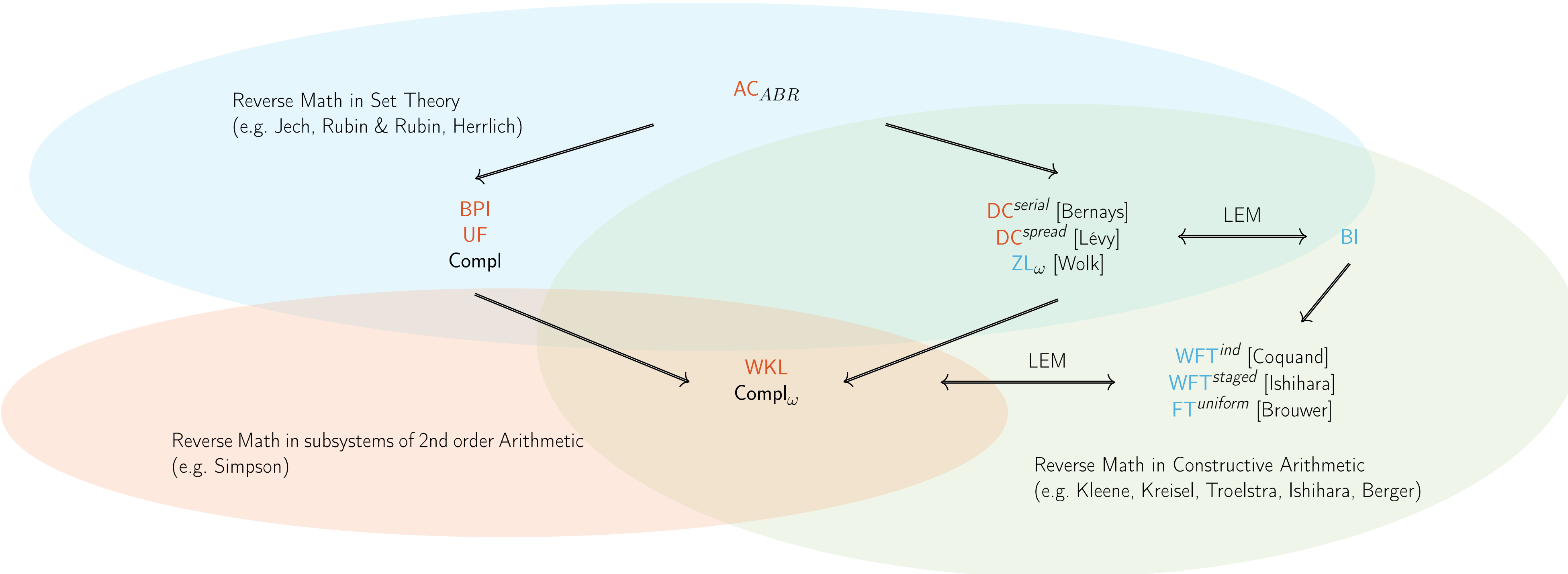
sometimes strictly weaker:

- axiom of **dependent** choice (DC), axiom of **countable** choice (AC_ω), Boolean prime ideal theorem (BPI), ultrafilter lemma (UF)

as well as variants in constructive mathematics, classically equivalent to choice or maximality principles:

- **bar induction**, its finite-branch version **fan theorem**, **update induction**, ...

Some standard results about the axiom of choice



BPI	=	Boolean Prime Ideal Theorem
UF	=	Ultrafilter Theorem
AC	=	Axiom of Choice
DC	=	Axiom of Dependent Choice
WKL	=	Weak Kőnig's Lemma

ZL_ω	=	Countable Zorn's Lemma
BI	=	Bar Induction
(W)FT	=	(Weak) Fan Theorem
Compl	=	Gödel's Completeness Theorem

Look at the axiom of choice and its variants from a **logical** and **computational** perspective

The logical perspective:

- The axiom of choice and their variants assert the existence of **ideal** objects from intensional properties of these objects
- See e.g. Coquand's program of reformulating standard mathematical statements using equivalent **inductive** properties to avoid the axiom of choice

$\hookrightarrow$ some variants can indeed be seen as **extensionality** principles

$\hookrightarrow$ other variants as **well-foundedness** of processes producing arbitrarily precise approximations of ideal objects

Look at the axiom of choice and its variants from a **logical** and **computational** perspective

The (long-term) computational perspective:

- Following Brouwer, we know from Kolmogorov, Kleene, Curry, Howard, and many other that **intuitionistic proofs** are **programs**
- We know from Griffin 1990 that also **classical** proofs are **programs**, though they use "goto"-like side effects
- We know from works in Paris that proofs by **forcing** are **programs**, using a memory
- Other effects such as Lisp's **quote** are also useful to compute with some axioms (see Krivine, Pédrot, ...)
- More generally, it can be shown (by abstract reasoning) that any consistent mathematical axiom has an underlying computational content
- What is the **computational** content of the axiom of choice and its variants (Krivine's research programme)?

Contribution I

- A classification of choice and bar induction principles by means of two **dual** forms, seen as **extensionality principles**, for T a predicate filtering the finite approximations of functions from A to B:

Generalised Bar Induction (GBI_{ABT})

$$\underbrace{T\ A\text{-}B\text{-}\mathit{barred}}_{observational} \implies \underbrace{T\ A\text{-}B\text{-}\mathit{inductively\ barred}}_{effective}$$

Generalised Dependent Choice (GDC_{ABT})

$$\underbrace{T\ \mathit{coinductively}\ A\text{-}B\text{-}\mathit{approximable}}_{effective} \implies \underbrace{T\ \mathit{has\ an}\ A\text{-}B\text{-}\mathit{choice\ function}}_{observational}$$

- such that:

$\mathsf{GBI}_{\mathbb{N}BT}$	denotes	BI_{BT}
$\mathsf{GBI}_{\mathbb{N}\mathbb{B}\mathsf{ool}T}$	denotes	FT_T
$\mathsf{GDC}_{A\mathbb{B}\mathsf{ool}T}$	denotes	BPI_{AT}
$\mathsf{GDC}_{\mathbb{N}BT}$	has the strength of	DC_{BR}
$\mathsf{GDC}_{\mathbb{N}\mathbb{B}\mathsf{ool}T}$	has the strength of	WKL_T
GDC_{ABT}	has the strength of	AC_{ABR} for T "split"

Contribution II

- A pair of **dual** maximality and well-foundedness principles, for T a predicate filtering the finite approximations of functions from A to B:

Generalised Update Induction ($\mathbf{GUI}_{ABT}$)

(generalising Berger's update induction to arbitrary cardinals)

if the upwards monotone closure of T is $\prec$-inductive, it contains all functions from A to B

$\exists$ Maximal Partial Choice Function ($\exists\mathbf{MPCF}_{ABT}$)

(a functional variant of Teichmüller-Tukey's lemma)

if the downwards closure by restriction of T is non empty, it has a $\prec$-maximal partial choice function from A to B

where $\alpha \prec \beta$ is the approximation order on partial functions from A to B.

- such that: when A is $\mathbb{N}$, or B is $\mathbb{B}$**ool**, or T is split, coinductive approximability implies the totality of the choice function, recovering the previous statements, and dually for barredness.
- and such that: Zorn's Lemma, Teichmüller-Tukey's lemma, and other maximality principles are particular instances of $\exists\mathbf{MPCF}$.

Outline

Part A is organised around the following oppositions

- ill-founded (choice axioms) / well-founded (bar induction axioms)
- extensional (ideal object) / intensional (processus)
- closed by sequential restriction (= tree) / closed by sequential extension (= monotony)
- binary branching (B is $\mathbb{B}$ool) / finite branching (B is finite) / arbitrary branching (B is arbitrary)

Part B moves to arbitrary cardinals, so as to capture BPI and full AC

- sequential (A countable) / unordered (A arbitrary)
- closed by unordered restriction (= ideal) / closed by unordered extension (= filter)

Part C moves to maximality and well-foundedness principles

Part A

The sequential case: Kőnig's lemma, fan theorem, dependent choice, bar induction

What is bar induction?

Let's consider first different ways to define well-foundedness

Trees (and their negative) as predicates

Let B be a domain and u ranges over the set B^* of finite sequences of elements of B. We write $\langle\rangle$ for the empty sequence and $u \star b$ for the extension with one element. For T a **predicate** on B^*, we define:

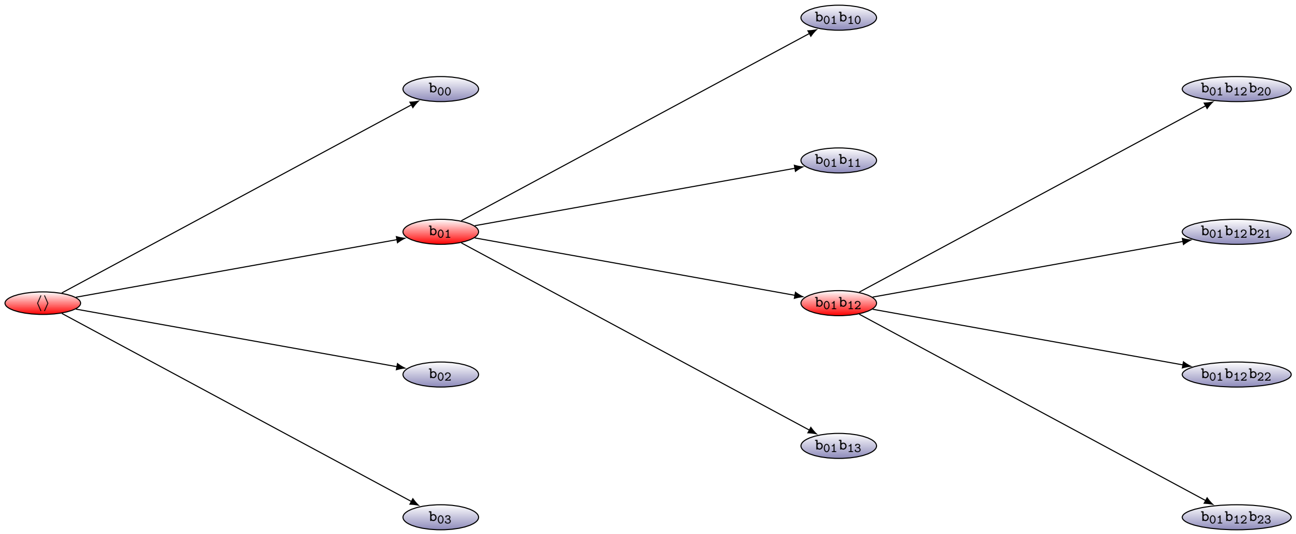
T is a tree	T is monotone
(closure under restriction)	(closure under extension)
$\forall u \, \forall a \, (u \star a \in T \Rightarrow u \in T)$	$\forall u \, \forall a \, (u \in T \Rightarrow u \star a \in T)$

Inductive characterisation of a well-founded tree-as-predicate

T *inductively well-founded* is short for inductively well-founded at $\langle\rangle \in A^*$

T *inductively well-founded at* u holds when:

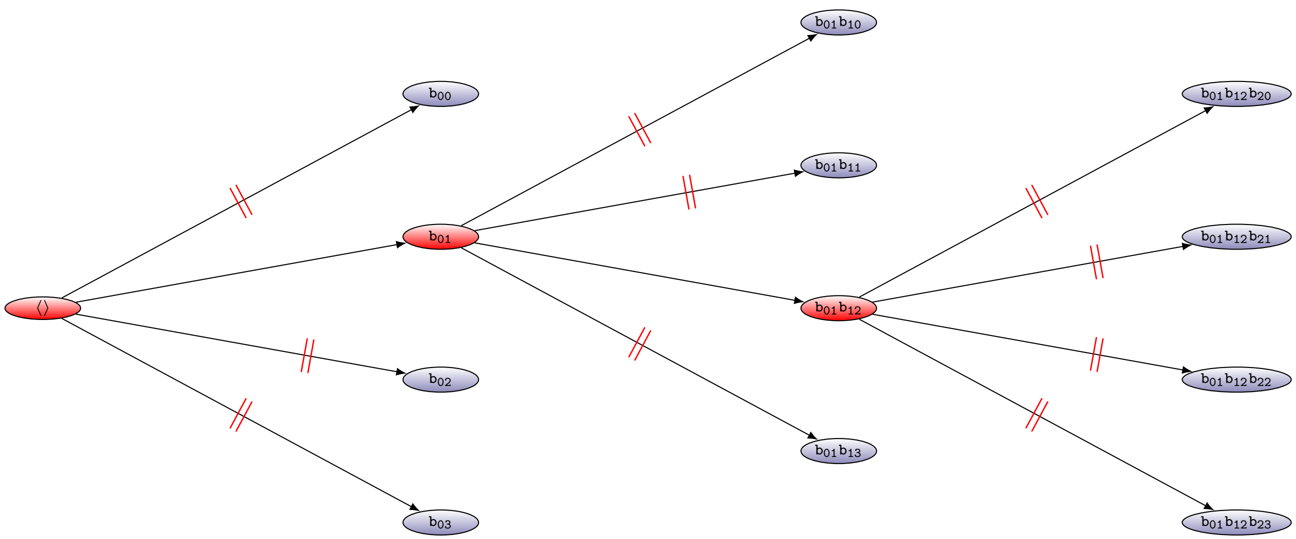
- $u \notin T$
- *or*, recursively, for all a, T is *inductively well-founded at* $u \star a$



Observational characterisation of a well-founded tree-as-predicate

T *observationally well-founded*

$$\forall \beta \in \mathbb{N} \to B.\ \exists n \in \mathbb{N}.\ \neg T(\beta_{|n})$$



Two characterisations of a well-founded tree-as-predicate

- From the **"effective"** representation of a well-founded tree we can always construct a predicate that is an **"observational"** representation of the tree
- To conversely obtain an effective representation of a tree T from its observational representation requires an axiom:

$$T \textit{ observationally well-founded} \implies T \textit{ inductively well-founded}$$

Bar Induction

If instead we build the **negative** of a tree-as-predicate and restate well-foundedness on the negative tree, one obtains bar induction:

- T *inductively well-founded* is the same as $\neg T$ *inductively barred*
- T *observationally well-founded* is the same $\neg T$ *barred*

- Bar Induction says that for a type B and a tree T,

$$\underbrace{T \text{ barred}}_{observational} \implies \underbrace{T \text{ inductively barred}}_{effective}$$

Dually: ill-foundedness

Dually, ill-foundedness of a tree T can be defined in different ways.

Let us concentrate on the finite-branching case. We have:

Effective view

$$T \text{ is staged infinite} \quad \triangleq \quad \forall n \, \exists u \, |u| = n \wedge u \in T$$

Observational view

$$T \text{ has an infinite branch} \quad \triangleq \quad \exists \alpha \, \forall u \leq \alpha \, T(u)$$

Kőnig's Lemma is a lemma that connects the two views when B is finite:

$$\mathsf{KL}_T \quad \triangleq \quad T \text{ is staged infinite} \Rightarrow T \text{ has an infinite branch}$$

Ill-foundedness, coinductively

Alternatively, by dualising the notion of inductively barred we get another **coinductive** definition of ill-foundedness, which we call productive. In full:

T productive is short for productive from $\langle\rangle \in B^*$

T productive from $u \in B^*$ holds when:

- u is in T
- *and*, recursively, there is $b \in B$ such that T is productive from $u \star b$

Relying on the notion of *inductively barred* and its dual, we obtain the following dual pair of choice and bar induction principles

Bar induction (BI_{BT})

T barred $\Rightarrow$ T inductively barred

Tree-Based Dependent Choice (DC_{BT}^{prod})

T productive $\Rightarrow$ T has an infinite branch

Recovering standard principles

$\mathsf{WKL}_T \iff \mathsf{DC}^{prod}_{\mathbb{B}\mathsf{ool}T}$ up to classical (actually co-intuitionistic) reasoning

$\mathsf{WFT}_T \iff \mathsf{BI}_{\mathbb{B}\mathsf{ool}T}$ up to intuitionistic reasoning

$\mathsf{DC}^{serial}_{BRb_0} \iff \mathsf{DC}^{prod}_{BR^{\triangleright}(b_0)}$

where

$$u \in R^{\triangleright}(b_0) \triangleq \text{case } u \text{ of } \left[\begin{array}{ll} \langle\rangle & \mapsto \top \\ b & \mapsto R(b_0, b) \\ u' \star b \star b' & \mapsto R(b, b') \end{array}\right]$$

$$\mathsf{DC}^{serial}_{BRb_0} \triangleq \forall b \exists b' \, R(b, b') \Rightarrow \exists \alpha \, (\alpha(0) = b_0 \wedge \forall n \, R(\alpha(n), \alpha(n+1)))$$

(one of the most standard statement of dependent choice)

Part B

Relaxing the sequentiality

Relaxing the sequentiality

Let A and B be domains. Let now use v to range over the set $(A \times B)^*$ of finite sequences of pairs of elements in A and B.

We say $(a, b) \in v$ if (a, b) is one of the components of v.

We write $v \leq v'$ if v is included in v' when seen as sets.

For $v \in (A \times B)^*$, we write $dom(v)$ for the set of a such that there is some b such that $(a, b) \in v$.

If $\alpha \in A \to B$, we write $v \subset \alpha$ and say that v is a finite approximation of α if $\alpha(a) = b$ for all $(a, b) \in v$.

Let T be a predicate on $(A \times B)^*$. We write $\downarrow T$ and $\uparrow T$ to mean the following inner and outer closures with respect to $\leq$:

$$v \in \downarrow T \triangleq \forall v' \leq v \; (v' \in T)$$

$$v \in \uparrow T \triangleq \exists v' \leq v \; (v' \in T)$$

Relaxing the sequentiality (effective view)

T inductively A-B-barred from $v \in (A \times B)^*$ holds when:

- v is in the outer closure of T
- *or*, recursively, there exists $a \notin dom(v)$ such that for all $b \in B$, T is inductively A-B-barred from $v \star (a, b)$

T coinductively A-B-approximable from $v \in (A \times B)^*$ holds when:

- v is in the inner closure of T
- *and*, recursively, for all $a \notin dom(v)$, there is $b \in B$ such that T is coinductively A-B-approximable from $v \star (a, b)$

Relaxing the sequentiality (observational view)

T A-B-barred if $\forall \alpha \in A \to B \ \exists v \subset \alpha \ (v \in T)$

T has an A-B-choice function if $\exists \alpha \in A \to B \ \forall v \subset \alpha \ (v \in T)$

This leads to the following generalisation

Generalised Bar Induction (GBI_{ABT})

$$\underbrace{T\ A\text{-}B\text{-}\textit{barred}}_{\textit{observational}} \implies \underbrace{T\ A\text{-}B\text{-}\textit{inductively barred}}_{\textit{effective}}$$

Generalised Dependent Choice (GDC_{ABT})

$$\underbrace{T\ \textit{coinductively}\ A\text{-}B\text{-}\textit{approximable}}_{\textit{effective}} \implies \underbrace{T\ \textit{has an}\ A\text{-}B\text{-}\textit{choice function}}_{\textit{observational}}$$

Results justifying the generalisation

$$\mathsf{GBI}_{\mathbb{N}BT} \iff \mathsf{BI}_{BT}$$

$$\mathsf{GDC}_{\mathbb{N}BT} \iff \mathsf{DC}_{BT}^{prod}$$

The Boolean Prime Ideal Theorem

The specialisation to $\mathbb{B}$**ool** of the generalisation also captures the Boolean Prime Ideal Theorem.

Let $(\mathcal{B}, \vee, \wedge, \bot, \top, \neg, \vdash)$ be a Boolean algebra and I an ideal on $\mathcal{B}$. We extend I on $(\mathcal{B} \times \mathbb{B}\mathsf{ool})^*$ by setting $u \in I^+$ if $(\bigvee_{(b,0)\in u} \neg b) \vee (\bigvee_{(b,1)\in u} b) \in I$. We have:

$$\mathsf{GDC}_{\mathcal{B}\mathbb{B}\mathsf{ool}I^+} \iff \mathsf{BPI}_{\mathcal{B},I}$$

The full axiom of choice

Let AC_{ABR} be $\forall a^A \, \exists b^B \, R(a,b) \Rightarrow \exists \alpha^{A \to B} \, \forall a^A \, R(a, \alpha(a))$

Define the *positive alignment* $R_\top$ of R by

$$R_\top \triangleq \lambda u. \, \forall (a,b) \in u \, R(a,b)$$

Then, AC_{ABR} arrives as the instance $\mathsf{GDC}_{ABR_\top}$

Strength of the generalisation

Without further restrictions, GDC and GBI are inconsistent:

- Take $A \triangleq \mathbb{N} \to \mathbb{B}\mathsf{ool}$
- Take $B \triangleq \mathbb{N}$
- Define T so that it constrains a choice function to be injective:

$$v \in T \triangleq \forall f f' n, ((f, n) \in v) \wedge ((f', n) \in v) \Rightarrow f = f'$$

Then, in the case of GDC, a coinductive A-B-approximation can always be found but an A-B-choice function would be an injective function from $\mathbb{N} \to \mathbb{B}\mathsf{ool}$ to $\mathbb{N}$, what is inconsistent.

A consistent restriction

A naive restriction is to require that:

- either A is countable
- or B is finite
- or T is **prime** (or split, atomic, or unary), meaning for all u and v:
  - in the ill-founded case $u \in T \wedge v \in T \Rightarrow u \cup v \in T$
  - in the barred case $u \cup v \in T \Rightarrow u \in T \vee v \in T$

The restriction preserves the previous instantiations and makes GDC equivalent to AC since it implies AC, and, conversely, each of its three restrictions is implied by a consequence of AC.

Dually for GBI.

Summary of main results regarding choice and bar induction

$\mathsf{AC}_{ABR} \triangleq \mathsf{GDC}_{ABR_\top}$

$\mathsf{BPI}_{AI} \triangleq \mathsf{GDC}_{A\mathbb{B}\mathsf{ool}\mathcal{T}_I}$
$\mathsf{Compl}^-_{A\mathcal{T}} \triangleq \mathsf{GDC}_{A\mathbb{B}\mathsf{ool}\mathcal{T}^C}$
$\mathsf{Compl}^+_{A\mathcal{T}} \triangleq \mathsf{GBI}_{A\mathbb{B}\mathsf{ool}\mathcal{T}^C}$

$\mathsf{DC}^{serial}_{BRb} \triangleq \mathsf{GDC}_{\mathbb{N}BR^{\triangleright}_{\top}}$
$\mathsf{DC}^{prod}_{BT} \triangleq \mathsf{GDC}_{\mathbb{N}BT}$
$\mathsf{ZL}_{\mathbb{N}R} \triangleq \mathsf{GBI}_{\mathbb{N}B(\neg R)^*_\perp}$
$\mathsf{BI}_{BT} \triangleq \mathsf{GBI}_{\mathbb{N}BT}$

$\mathsf{WKL}^{prod}_{BT} \triangleq \mathsf{GDC}_{\mathbb{N}\mathbb{B}\mathsf{ool}T}$
$\mathsf{WFT}^{ind}_{BT} \triangleq \mathsf{GBI}_{\mathbb{N}\mathbb{B}\mathsf{ool}T}$

AC	=	Axiom of Choice
DC	=	Axiom of Dependent Choice
BPI	=	Boolean Prime Ideal Theorem
Compl^-	=	Completeness (consistent ⇒ model)
WKL	=	Weak Kőnig's Lemma

Compl^+	=	Completeness (valid ⇒ provable)
ZL	=	Zorn's Lemma
BI	=	Bar Induction
WFT	=	Weak Fan Theorem

Part C

Maximality and well-foundedness principles

A first solution to the inconsistency of the general form of GDC: requiring only a maximal partial function

Generalised Maximal Dependent Choice

$$\underbrace{T \text{ coinductively } A\text{-}B\text{-approximable}}_{\textit{effective}} \implies \underbrace{T \text{ has a } \textbf{maximal partial } A\text{-}B\text{-choice function}}_{\textit{observational}}$$

However, approximability happens to be a useless hypothesis, so we can remove it.

∃ Maximal Partial Choice Function ($\exists\mathbf{MPCF}_{ABT}$)

∃ *Maximal Partial Choice* Function

T *non-empty* $\Longrightarrow$ T *has a* ***maximal partial*** A-B-*choice function*

This happens to be very close to Teichmüller-Tukey Lemma and its contrapositive to Berger's update induction.

Different possible definitions of a partial function $\alpha : A \rightharpoonup B$

(non constructively equivalent though)

- a (non-necessarily left-total) functional relation (leading to $\exists\mathbf{MPCF}^{rel}$)

- a total function to a codomain extended with an element $\bot$ standing for undefinedness (leading to $\exists\mathbf{MPCF}^{dec}$)

Then, we can define in each case a relation $\beta \prec \alpha$ standing for β is strictly more defined than α

Teichmüller-Tukey Lemma

Let T be a predicate over A^*. We define its powerset closure by downwards restriction $\langle T \rangle$ as:

$$\langle T \rangle \quad \triangleq \quad \lambda\alpha^{\mathcal{P}(A)}.\forall u^{A^*}(u \subset \alpha \to u \in T)$$

Then, we say that a predicate P over predicates over A is of **finite character** if there is T such that $P = \langle T \rangle$.

Then, we can conversely rebuild T from $\langle T \rangle$ by setting

$$\begin{array}{lcl} \hat{u} & \triangleq & \lambda x^A.\ x \in u \\ \lfloor P \rfloor & \triangleq & \lambda u^{A^*}.\ \hat{u} \in P \end{array}$$

so that $T = \lfloor \langle T \rangle \rfloor$ and so that P is of finite character iff $P = \langle \lfloor P \rfloor \rangle$.

Teichmüller-Tukey Lemma

Teichmüller-Tukey $\mathbf{TTL}$ is the statement that any non-empty predicate of finite character (thus derived from some $T : \mathcal{P}(A)$) has a maximal element with respect to inclusion.

We have:

$$\mathbf{TTL}_{AT} \simeq \exists\mathbf{MPCF}^{rel}_{A1(T\circ\pi_1)}$$

$$\mathbf{TTL}_{(A\times B)T} \simeq \exists\mathbf{MPCF}^{rel}_{ABT}$$

And, incidentally, for an appropriate construction $\mathbf{C}_{<E}$:

$$\mathbf{TTL}_{A\mathbf{C}_{<E}} \iff \mathsf{ZL}_{A<E}$$

$$\mathbf{TTL}_{AT} \iff \mathsf{ZL}_{\mathcal{P}(A)\subset\langle T\rangle}$$

$\exists\mathbf{MPCF}^{dec}_{\mathbb{N}BT}$ is the contrapositive of Berger's update induction, and conversely, update induction can be generalised to arbitrary domains

P is of **finite character** over partial functions from $\mathbb{N}$ to B is the same as $\neg P$ **open predicate** in Berger's sense. This leads to the following:

Generalised Update Induction ($\mathbf{GUI}^{dec}_{ABT}$)

if the upwards monotone closure of T is $\prec$-inductive, it contains all partial functions from A to B

where the upwards monotone closure of T is:

$$\langle T\rangle^{\circ} \triangleq \lambda\alpha^{\mathcal{P}(A\times B)}.\exists u^{(A\times B)^*}(u \subset \alpha \wedge u \in T)$$

Clarifying the whole picture around $\exists\mathbf{MPCF}$, $\mathbf{TTL}$, their relational or decidable versions, their sequential version, as well as the contrapositive picture around $\mathbf{GUI}$, is however left for future work...