Università di Genova | **DIMA** DIPARTIMENTO DI MATEMATICA

Ultracompletions

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Joint work in progress with
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Foundations of Mathematics
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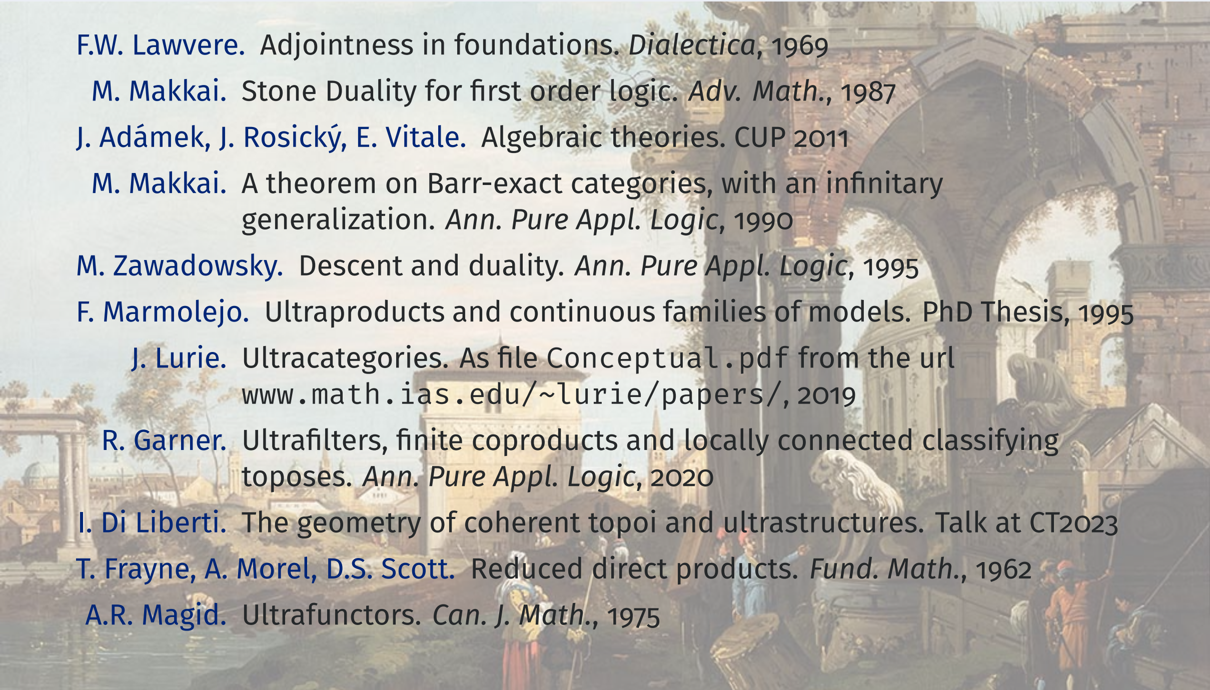
Prologue



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Prologue

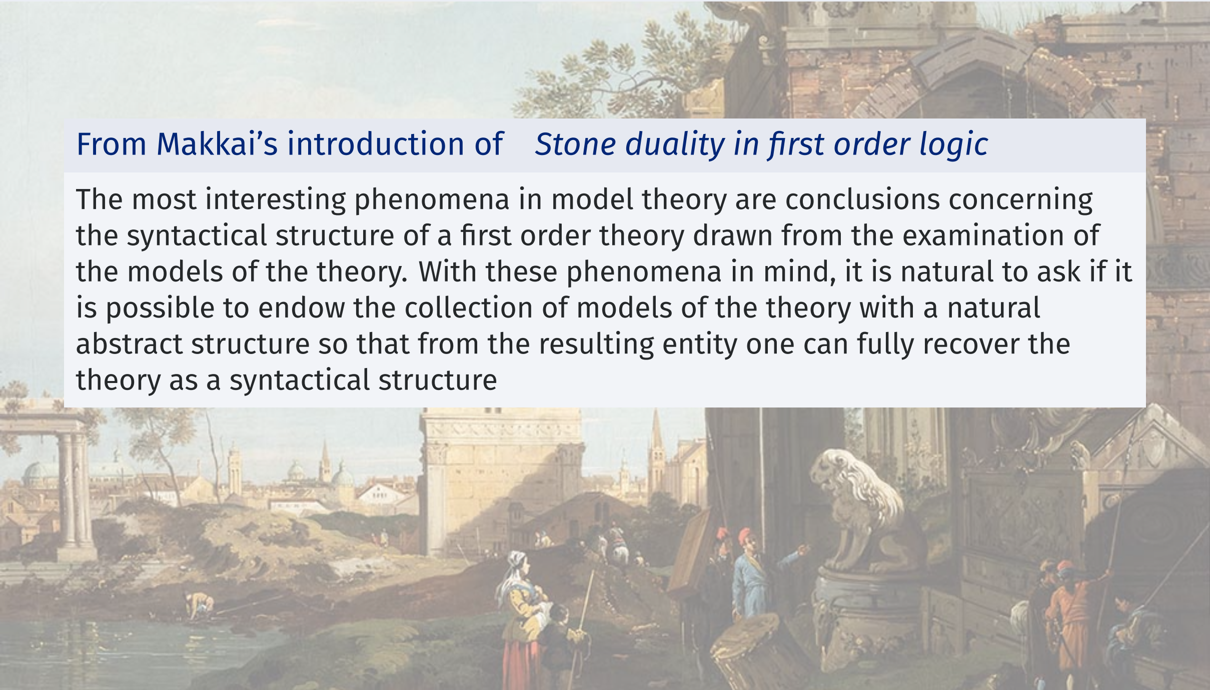
From Lawvere’s commentary for the TAC reprint of *Adjointness in foundations*

Local Galois connections (in algebraic geometry, model theory, linear algebra, etc.) are globalized into functors, such as Spec, carrying much more information. Also, “theories” (even when presented symbolically) are viewed explicitly as categories; so are the background universes of sets that serve as the recipients for models. (Models themselves are functors, hence preserve the fundamental operation of substitution/composition in terms of which the other logical operations can be characterized as local adjoints.)[...]

Inspired by Läuchli’s 1967 success in finding a completeness theorem for Heyting predicate calculus lurking in the category of ordinary permutations, I presented, at the 1967 AMS Los Angeles Symposium on Set Theory, a common functorization of several geometrical structures, including such proof-theoretic structures.

As Hyperdoctrines, those structures are described in the Proceedings of the AMS New York Symposium XVII [“Equality in hypedoctrines and the comprehension schema as adjoint functors”]

Prologue



From Makkai’s introduction of *Stone duality in first order logic*

> The most interesting phenomena in model theory are conclusions concerning the syntactical structure of a first order theory drawn from the examination of the models of the theory. With these phenomena in mind, it is natural to ask if it is possible to endow the collection of models of the theory with a natural abstract structure so that from the resulting entity one can fully recover the theory as a syntactical structure

Ultrafilters

Definition

An **ultrafilter** $\mathcal{U}$ on a set I is a collection of subsets of I such that

- $\emptyset \notin \mathcal{U}$
- if $A \notin \mathcal{U}$ then $I \setminus A \in \mathcal{U}$
- if $A \in \mathcal{U}$ and $A \subseteq B$ then $B \in \mathcal{U}$
- if $A_1 \in \mathcal{U}$ and $A_2 \in \mathcal{U}$ then $A_1 \cap A_2 \in \mathcal{U}$

Ultrafilters

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An **ultrafilter** $\mathcal{U}$ on a set I is a collection of subsets of I such that

$$\left.\begin{array}{l}
\blacktriangleright\ \emptyset \notin \mathcal{U} \\
\blacktriangleright\ \text{if } A \notin \mathcal{U} \text{ then } I \setminus A \in \mathcal{U} \\
\blacktriangleright\ \text{if } A \in \mathcal{U} \text{ and } A \subseteq B \text{ then } B \in \mathcal{U} \\
\blacktriangleright\ \text{if } A_1 \in \mathcal{U} \text{ and } A_2 \in \mathcal{U} \text{ then } A_1 \cap A_2 \in \mathcal{U}
\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}
\blacktriangleright\ I \in \mathcal{U} \\
\blacktriangleright\ \emptyset \notin \mathcal{U} \\
\blacktriangleright\ (A_1 \in \mathcal{U} \text{ and } A_2 \in \mathcal{U}) \text{ iff } A_1 \cap A_2 \in \mathcal{U} \\
\blacktriangleright\ (A_1 \in \mathcal{U} \text{ or } A_2 \in \mathcal{U}) \text{ iff } A_1 \cup A_2 \in \mathcal{U}
\end{array}\right.$$

In other words $\mathbf{2}^I \xrightarrow{\mathrm{ch}(\mathcal{U})} \mathbf{2}$ is a lattice homomorphism

Ultraproducts

Definition

The $\mathcal{U}$-**ultraproduct** of the family of sets $S_\bullet = (S_i)_{i\in I}$ is

$$\prod_{\mathcal{U}}(S_\bullet) \overset{\mathrm{df}}{=} \left[\bigcup_{U\in\mathcal{U}} \left(\prod_{i\in U} S_i\right)\right] /_{\sim_{\mathcal{U}}}$$

where $\sigma \sim_{\mathcal{U}} \sigma'$ means that $[\![\sigma = \sigma']\!] \overset{\mathrm{df}}{=} \left\{ i \in \mathrm{dom}(\sigma) \cap \mathrm{dom}(\sigma') \,\middle|\, \sigma(i) = \sigma'(i) \right\} \in \mathcal{U}$

It is the colimit of the filtered diagram

$$\begin{tikzcd}
 & & \prod_{i\in U'} S_i & \\
 & \prod_{i\in U} S_i \arrow[ur, "\mathrm{p}_{U'\subseteq U}"] \arrow[drr, "\mathrm{p}_{U\cap V\subseteq U}"'] & & \\
 & & & \prod_{i\in U\cap V} S_i \\
\prod_{i\in I} S_i \arrow[uur, "\mathrm{p}_{U\subseteq I}"] \arrow[drr, "\mathrm{p}_{V\subseteq I}"'] & & & \\
 & & \prod_{i\in V} S_i \arrow[uur, "\mathrm{p}_{U\cap V\subseteq V}"] &
\end{tikzcd}$$

Ultraproducts

Definition

The $\mathcal{U}$-**ultraproduct** of the family of sets $S_\bullet = (S_i)_{i\in I}$ is

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It is the colimit of the filtered diagram

$$\begin{tikzcd}
 & & \prod_{i\in U'} S_i \arrow[rrdd] & & \\
 & \prod_{i\in U} S_i \arrow[ur, "\mathrm{p}_{U'\subseteq U}"] \arrow[rrd, "\mathrm{p}_{U\cap V\subseteq U}"'] \arrow[rrrd] & & & \\
 & & & \prod_{i\in U\cap V} S_i \arrow[r] & \prod_{\mathcal{U}} S_\bullet \\
\prod_{i\in I} S_i \arrow[uur, "\mathrm{p}_{U\subseteq I}"] \arrow[ddrr, phantom] \arrow[rrrru] \arrow[drr, "\mathrm{p}_{V\subseteq I}"'] & & & & \\
 & & \prod_{i\in V} S_i \arrow[uur, "\mathrm{p}_{U\cap V\subseteq V}"] \arrow[rruu] & &
\end{tikzcd}$$

Ultraproducts

It gives rise to a functor

$$\begin{tikzcd}
\mathbf{Set}^I \arrow[r, "\prod_{\mathcal{U}}"] & \mathbf{Set} \\
S_\bullet = (S_i)_{i\in I} \arrow[r, mapsto] & \operatorname{colim}_{U\in\mathcal{U}} \prod_{i\in U} S_i
\end{tikzcd}$$

which preserves finite limits

Standard properties of ultraproducts

- $S \rightarrowtail \prod_{\mathcal{U}}(\hat{S})$ for a constant family $\hat{S} = (S)_{i\in I}$
- $\prod_{i^\in}(S_\bullet) \xrightarrow{\sim} S_i$ for a principal ultrafilter $i^\in \overset{\text{df}}{=} \left\{U \subseteq I \,\middle|\, i \in U\right\}$
- $\prod_{\sum_{\mathcal{U}} \mathcal{V}_\bullet}(S_*) \xrightarrow{\sim} \prod_{\mathcal{U}}(\prod_{\mathcal{V}_\bullet}(S_*))$ for $\sum_{\mathcal{U}} \mathcal{V}_\bullet$ the ultrafilter on $\sum_{i\in I} J_i$ determined by

$$W \in \textstyle\sum_{\mathcal{U}} \mathcal{V}_\bullet \overset{\text{df}}{\Leftrightarrow} \left\{i \in I \,\middle|\, \left\{j \in J_i \,\middle|\, (i,j) \in W\right\} \in \mathcal{V}_j\right\} \in \mathcal{U}$$

- $\prod_{\mathcal{U}_J}(S_\bullet) \xrightarrow{\sim} \prod_{\mathcal{U}}(S_\bullet)$ for $I \subseteq J$ and $\mathcal{U}_J \overset{\text{df}}{=} \left\{U \subseteq J \,\middle|\, U \cap I \in \mathcal{U}\right\}$

∴ $\prod_{\mathcal{U}}(S_\bullet) \xrightarrow{\sim} \prod_{\mathcal{U}\restriction_U}(S_\bullet)$ for $U \in \mathcal{U}$ and $\mathcal{U}\restriction_U \overset{\text{df}}{=} \left\{V \subseteq U \,\middle|\, V \in \mathcal{U}\right\}$ since $\left(\mathcal{U}\restriction_U\right)_I = \mathcal{U}$

Ultraproducts

It gives rise to a functor

$$\begin{tikzcd}
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- ▶ $\prod_{\sum_{\mathfrak{U}} \mathcal{V}_\bullet}\big((S_j)_{(i,j)\in\sum_i J_i}\big) \xrightarrow{\sim} \prod_{\mathfrak{U}}\big(\prod_{\mathcal{V}_i}((S_j)_{j\in J_i})\big)$ for $\sum_{\mathfrak{U}} \mathcal{V}_\bullet$ the ultrafilter on $\sum_{i\in I} J_i$ dete

$$W \in \sum_{\mathfrak{U}} \mathcal{V}_\bullet \overset{\text{df}}{\Leftrightarrow} \Big\{i \in I \,\Big|\, \{j \in J_i \mid (i,j) \in W\} \in \mathcal{V}_j\Big\} \in \mathfrak{U}$$

- ▶ $\prod_{\mathfrak{U}_J}\big((S_j)_{j\in J}\big) \xrightarrow{\sim} \prod_{\mathfrak{U}}\big((S_i)_{i\in I}\big)$ for $I \subseteq J$ and $\mathfrak{U}_J \overset{\text{df}}{=} \{U \subseteq J \mid U \cap I \in \mathfrak{U}\}$
- ∴ $\prod_{\mathfrak{U}}\big((S_i)_{i\in I}\big) \xrightarrow{\sim} \prod_{\mathfrak{U}\restriction_U}\big((S_i)_{i\in U}\big)$ for $U \in \mathfrak{U}$ and $\mathfrak{U}\restriction_U \overset{\text{df}}{=} \{V \subseteq U \mid V \in \mathfrak{U}\}$ since (

Ultraproducts

It gives rise to a functor

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which preserves finite limits

Theorem (Łoš, 1955)

Suppose $(\mathfrak{S}_i)_{i\in I}$ is a family of interpretation of the first order language $\mathcal{L}$
Let $\phi(x_1,\dots,x_n)$ be any formula in $\mathcal{L}$ and let $[\sigma_1],\dots,[\sigma_n] \in \prod_{\mathfrak{U}}(S_\bullet)$
Then

$$\prod_{\mathfrak{U}}\mathfrak{S}_\bullet \models \phi([\sigma_1],\dots,[\sigma_n]) \text{ if and only if } [\![\phi(\sigma_1(i),\dots,\sigma_n(i))]\!] \in \mathfrak{U}$$

where $[\![\phi(\sigma_1(i),\dots,\sigma_n(i))]\!] \overset{\text{df}}{=} \left\{i \in \bigcap_{l\le n} \mathrm{dom}(\sigma_l) \;\middle|\; \mathfrak{S}_i \models \phi(\sigma_1(i),\dots,\sigma_n(i))\right\}$

Ultraproducts

It gives rise to a functor

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which preserves finite limits

Theorem (Łoš)

The functor $\prod_{\mathcal{U}} \colon \mathbf{Set}^I \longrightarrow \mathbf{Set}$ is part of a homomorphism of first order hyperdoctrines

$$\begin{tikzcd} (\mathrm{Sub}_{\mathbf{Set}^I})^I \arrow[r, "\prod_{\mathcal{U}}"] \arrow[d, "\mathrm{cod}^I"'] & \mathrm{Sub}_{\mathbf{Set}} \arrow[d, "\mathrm{cod}"] \\ \mathbf{Set}^I \arrow[r, "\prod_{\mathcal{U}}"] & \mathbf{Set} \end{tikzcd}$$

Ultraproducts

Theorem (Łoš)

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$$\begin{tikzcd}
(\mathrm{Sub}_{\mathbf{Set}^I})^I \arrow[r, "\prod_{\mathcal{U}}"] \arrow[d, "\mathrm{cod}^I"'] & \mathrm{Sub}_{\mathbf{Set}} \arrow[d, "\mathrm{cod}"] \\
\mathbf{Set}^I \arrow[r, "\prod_{\mathcal{U}}"] & \mathbf{Set}
\end{tikzcd}$$

Hence $\mathbf{Set}^I \xrightarrow{\prod_{\mathcal{U}}} \mathbf{Set}$ is a homomorphism of pretoposes

It extends to $\mathbf{Pret}(\mathbf{T},\mathbf{Set})^I \xrightarrow{\prod_{\mathcal{U}}} \mathbf{Pret}(\mathbf{T},\mathbf{Set})$ for $\mathbf{T}$ a pretopos

Makkai's ultracategories

Definition

An **ultracategory** (in the sense of Makkai) is a category $\mathbf{C}$ endowed with

- a functor $[\mathfrak{U}]\colon \mathbf{C}^I \longrightarrow \mathbf{C}$ for each I and ultrafilter $\mathfrak{U}$ on I
- a natural transformation

$$\begin{tikzcd}
\mathbf{Uld}(\Gamma, \mathbf{C}) \arrow[r, bend left, "\mathrm{Ev}_n", ""{name=U, below}] \arrow[r, bend right, "\mathrm{Ev}_m"', ""{name=D, above}] & \mathbf{C} \arrow[Rightarrow, from=U, to=D, "\delta"']
\end{tikzcd}$$

for each ultradiagram Γ and ultramorphism δ between nodes n and m

From Marmolejo's PhD thesis, 1995

There is very short supply of canonical maps in or out of an ultraproduct (as oppose to an honest limit or colimit). Ultramorphisms try to fix this

Makkai's ultracategories

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$$\begin{tikzcd}
\mathbf{Uld}(\Gamma, \mathbf{C}) \arrow[r, shift left=3, "\mathrm{Ev}_n"] \arrow[r, shift right=3, "\mathrm{Ev}_m"'] \arrow[r, phantom, "\delta \Downarrow \cdot"] & \mathbf{C}
\end{tikzcd}$$

for each ultradiagram Γ and ultramorphism δ between nodes n and m

From Garner's APAL paper, 2020

An ultracategory is a category endowed with abstract ultraproduct functors together with interpretations for any "definable map between ultraproducts"

Makkai's ultracategories

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- a natural transformation

$$\begin{tikzcd}
\mathbf{Uld}(\Gamma, \mathbf{C}) \arrow[r, bend left, "\mathrm{Ev}_n", ""{name=A, below}] \arrow[r, bend right, "\mathrm{Ev}_m"', ""{name=B, above}] & \mathbf{C} \arrow[Rightarrow, from=A, to=B, "\delta", "\cdot"']
\end{tikzcd}$$

for each ultradiagram Γ and ultramorphism δ between nodes n and m

From Di Liberti's talk at CT2023

[N]one of these notions appears definitive when read or encountered for the first time for several reasons. The main one being that the definition of ultracategory is in both cases very heavy, and comes together with axioms whose choice seems quite arbitrary; and indeed the two authors make different choices

The ultracompletion $\mathrm{Ult}(\mathbf{C})$ of a category $\mathbf{C}$

An object is $(I,\mathcal{U},(c_i)_{i\in I})$ where

- $\mathcal{U}$ is an ultrafilter I
- $(c_i)_{i\in I}$ is an I-indexed family of objects in $\mathbf{C}$

An arrow is $[V,f,(g_v)_{v\in V}]\colon (I,\mathcal{U},(c_i)_{i\in I})\longrightarrow(J,\mathcal{V},(d_j)_{j\in J})$ a $\sim_{\mathcal{V}}$-equivalence class of

- a set $V\in\mathcal{V}$
- a function $f\colon V\longrightarrow I$ such that $f^{-1}(U)\in\mathcal{V}$ for each $U\in\mathcal{U}$
- a family $(g_v\colon c_{f(v)}\longrightarrow d_v)_{v\in V}$ of arrows in $\mathbf{C}$

and $(V,f,(g_v)_{v\in V})\sim_{\mathcal{V}}(V',f',(g'_v)_{v\in V'})$ when $[\![g_v=g'_v]\!]\in\mathcal{V}$

Composition is componentwise

Proposition

$\mathrm{Ult}(\mathbf{C})$ is equivalent to $\left(\mathcal{U}\mathcal{F}_{\mathrm{Fam}(\mathbf{C}^{\mathrm{op}})}\right)^{\mathrm{op}}$ [Garner, 2020]

“Ultraproduct” functors in $\mathrm{Ult}(\mathbf{C})$

Let $\mathcal{U}$ be an ultrafilter on I

$$\begin{tikzcd}
\mathrm{Ult}(\mathbf{C})^I \arrow[r, "{[\mathcal{U}]}"] & \mathrm{Ult}(\mathbf{C}) \\
\left(J_i, \mathcal{V}_i, (d_j)_{j\in J_i}\right)_{i\in I} \arrow[r, mapsto] \arrow[d, dash] & \left(\sum_{i\in I} J_i, \sum_{\mathcal{U}} \mathcal{V}_\bullet, (d_j)_{(i,j)\in\sum_{i\in I} J_i}\right) \arrow[d, dash] \\
\left[V_i, f_i, (g_w)_{w\in W_i}\right]_{i\in I} \arrow[r, mapsto] \arrow[d] & \left[\sum_{i\in I} V_i, \sum_{i\in I} f_i, (g_w)_{(i,w)\in\sum_{i\in I} W_i}\right] \arrow[d] \\
\left(L_i, \mathcal{W}_i, (e_l)_{l\in L_i}\right) \arrow[r, mapsto] & \left(\sum_{i\in I} L_i, \sum_{\mathcal{U}} \mathcal{W}_\bullet, (e_l)_{(i,l)\in\sum_{i\in I} L_i}\right)
\end{tikzcd}$$

where $\sum_{\mathcal{U}} \mathcal{V}_\bullet$ is the ultrafilter on $\sum_{i\in I} J_i$

$$W \in \sum_{\mathcal{U}} \mathcal{V}_\bullet \overset{\text{df}}{\Leftrightarrow} \left\{ i \in I \,\middle|\, \left\{ j \in J_i \,\middle|\, (i,j) \in W \right\} \in \mathcal{V}_j \right\} \in \mathcal{U}$$

$$\begin{tikzcd}
\mathbf{2}^{\sum_{i\in I} J_i} \arrow[r, "\mathbf{2}^{\mathrm{inj}_i}"'] & \mathbf{2}^{J_i} \arrow[r, "\mathrm{ch}\left(\mathcal{V}_i\right)"'] & \mathbf{2}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{2}^{\sum_{i\in I} J_i} \arrow[r, "\langle \mathrm{ch}\left(\mathcal{V}_i\right)\mathbf{2}^{\mathrm{inj}_i}\rangle_{i\in I}"] \arrow[rr, bend right, "\mathrm{ch}\left(\sum_{\mathcal{U}} \mathcal{V}_\bullet\right)"'] & \mathbf{2}^I \arrow[r, "\mathrm{ch}\left(\mathcal{U}\right)"] & \mathbf{2}
\end{tikzcd}$$

“Ultraproduct” functors in $\mathrm{Ult}(\mathbf{C})$

Let $\mathcal{U}$ be an ultrafilter on I

$$\begin{tikzcd}
\mathrm{Ult}(\mathbf{C})^I \arrow[rr, "{[\mathcal{U}]}"] & & \mathrm{Ult}(\mathbf{C}) \\
\left(J_i, \mathcal{V}_i, (d_j)_{j\in J_i}\right)_{i\in I} \arrow[rr, mapsto] \arrow[d, dash] & & \left(\sum_{i\in I} J_i, \sum_{\mathcal{U}} \mathcal{V}_{\bullet}, (d_j)_{(i,j)\in\sum_{i\in I} J_i}\right) \arrow[d, dash] \\
\left[V_i, f_i, (g_w)_{w\in W_i}\right]_{i\in I} \arrow[rr, mapsto] \arrow[d] & & \left[\sum_{i\in I} V_i, \sum_{i\in I} f_i, (g_w)_{(i,w)\in\sum_{i\in I} W_i}\right] \arrow[d] \\
\left(L_i, \mathcal{W}_i, (e_l)_{l\in L_i}\right) \arrow[rr, mapsto] & & \left(\sum_{i\in I} L_i, \sum_{\mathcal{U}} \mathcal{W}_{\bullet}, (e_l)_{(i,l)\in\sum_{i\in I} L_i}\right)
\end{tikzcd}$$

Proposition

The hom-set $\mathrm{Ult}(\mathbf{C})\left((K, \mathcal{W}, (c_k)_{k\in K}), [\mathcal{U}]\left(J_i, \mathcal{V}_i, (d_j)_{j\in J_i}\right)_{i\in I}\right)$ is in bijection with

$$\prod_{\mathcal{U}}\left(\mathrm{Ult}(\mathbf{C})\left((K, \mathcal{W}, (c_k)_{k\in K}), \left(J_i, \mathcal{V}_i, (d_j)_{j\in J_i}\right)_{i\in I}\right)\right)$$

The pseudomonad on the ultracompletion

Theorem

The 2-functor $\mathrm{Ult}\colon \mathbf{Cat} \longrightarrow \mathbf{Cat}$ is part of a colax idempotent pseudomonad

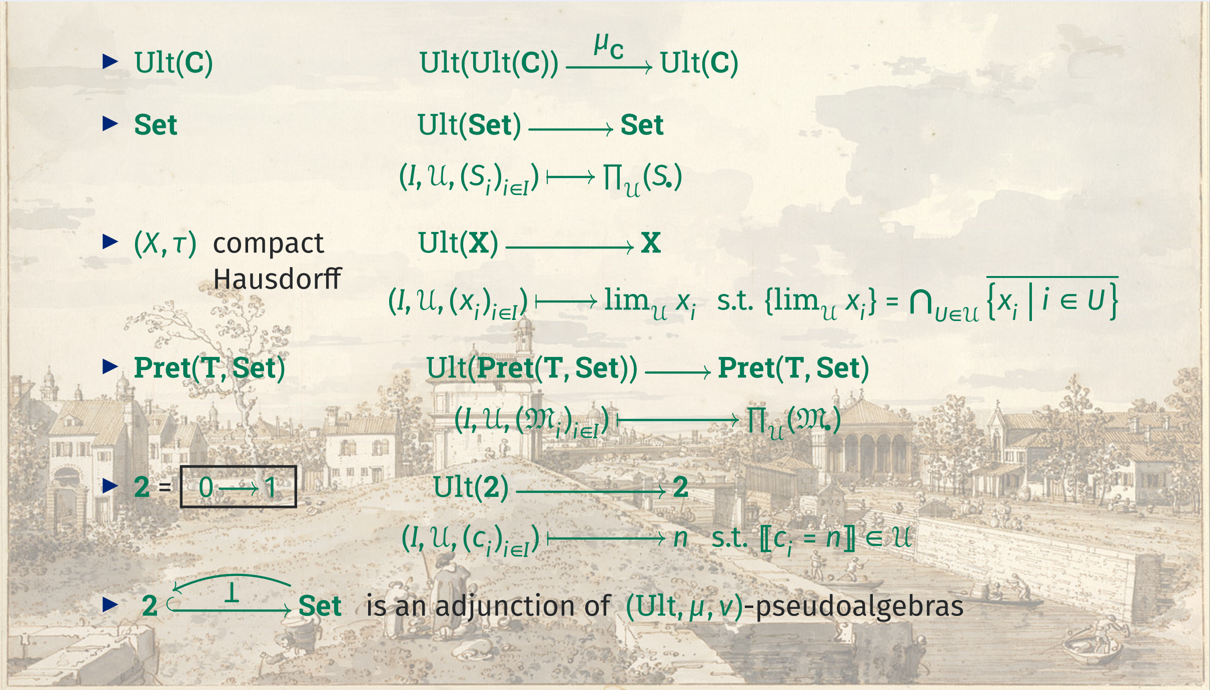
Proof

$$\begin{tikzcd}
\mathrm{Ult}(\mathrm{Ult}(\mathbf{C})) \arrow[r, "\mu_{\mathbf{C}}"] & \mathrm{Ult}(\mathbf{C}) \\
\Big(I, \mathcal{U}, (J_i, \mathcal{V}_i, (c_j)_{j\in J_i})_{i\in I}\Big) \arrow[r, mapsto] & \Big(\Sigma_{i\in I} J_i, \Sigma_{\mathcal{U}} \mathcal{V}_\bullet, (c_j)_{(i,j)\in\Sigma_{i\in I} J_i}\Big) = [\mathcal{U}]\Big(J_i, \mathcal{V}_i, (c_j)_{j\in J_i}\Big)_{i\in I}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "\nu_{\mathbf{C}}"] & \mathrm{Ult}(\mathbf{C}) \\
c \arrow[r, mapsto] & (1, \{1\}, (c))
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & (I, \mathcal{U}, ((1, \{1\}, (c_i)))_{i\in I}) \\
(I, \mathcal{U}, (c_i)_{i\in I}) \arrow[urrr, mapsto, bend left] \arrow[drrr, mapsto, bend right] & \mathrm{Ult}(\mathbf{C}) \arrow[r, bend left, "\mathrm{Ult}(\nu_{\mathbf{C}})"] \arrow[r, bend right, "\nu_{\mathrm{Ult}(\mathbf{C})}"'] \arrow[r, phantom, "\lambda_{\mathbf{C}}\Uparrow"] & \mathrm{Ult}(\mathrm{Ult}(\mathbf{C})) & \\
 & & & (1, \{1\}, (I, \mathcal{U}, (c_i)_{i\in I})) \arrow[uu, "\Big[I, !, [, i!, (\mathrm{id}_{c_i})]_{i\in I}\Big]"']
\end{tikzcd}$$

Examples of (Ult, μ, ν)-pseudoalgebras



- $\mathrm{Ult}(\mathbf{C})$ $\qquad$ $\mathrm{Ult}(\mathrm{Ult}(\mathbf{C})) \xrightarrow{\mu_{\mathbf{C}}} \mathrm{Ult}(\mathbf{C})$
- **Set** $\qquad$ $\mathrm{Ult}(\mathbf{Set}) \longrightarrow \mathbf{Set}$

  $(I, \mathcal{U}, (S_i)_{i \in I}) \longmapsto \prod_{\mathcal{U}}(S_\bullet)$
- (X, τ) compact Hausdorff $\qquad$ $\mathrm{Ult}(\mathbf{X}) \longrightarrow \mathbf{X}$

  $(I, \mathcal{U}, (x_i)_{i \in I}) \longmapsto \lim_{\mathcal{U}} x_i$ s.t. $\{\lim_{\mathcal{U}} x_i\} = \bigcap_{U \in \mathcal{U}} \overline{\{x_i \mid i \in U\}}$
- $\mathbf{Pret}(\mathbf{T}, \mathbf{Set})$ $\qquad$ $\mathrm{Ult}(\mathbf{Pret}(\mathbf{T}, \mathbf{Set})) \longrightarrow \mathbf{Pret}(\mathbf{T}, \mathbf{Set})$

  $(I, \mathcal{U}, (\mathfrak{M}_i)_{i \in I}) \longmapsto \prod_{\mathcal{U}}(\mathfrak{M}_\bullet)$
- $\mathbf{2} = \boxed{0 \longrightarrow 1}$ $\qquad$ $\mathrm{Ult}(\mathbf{2}) \longrightarrow \mathbf{2}$

  $(I, \mathcal{U}, (c_i)_{i \in I}) \longmapsto n$ s.t. $[\![c_i = n]\!] \in \mathcal{U}$
- $$\begin{tikzcd} \mathbf{2} \arrow[r, hook] \arrow[r, phantom, bend left=30, "\perp"] & \mathbf{Set} \arrow[l, bend right=40] \end{tikzcd}$$ is an adjunction of (Ult, μ, ν)-pseudoalgebras

Examples of Ult-pseudoalgebras

- $\mathrm{Ult}(\mathbf{C})$

  $$\mathrm{Ult}(\mathrm{Ult}(\mathbf{C})) \xrightarrow{\mu_{\mathbf{C}}} \mathrm{Ult}(\mathbf{C})$$

- $\mathbf{Set}$

  $$\mathrm{Ult}(\mathbf{Set}) \longrightarrow \mathbf{Set}$$

  $$(I, \mathcal{U}, (S_i)_{i \in I}) \longmapsto \prod_{\mathcal{U}}(S_\bullet)$$

- (X, τ) compact Hausdorff

  $$\mathrm{Ult}(\mathbf{X}) \longrightarrow \mathbf{X}$$

  $$(I, \mathcal{U}, (x_i)_{i \in I}) \longmapsto \lim_{\mathcal{U}} x_i \quad \text{s.t. } \{\lim_{\mathcal{U}} x_i\} = \bigcap_{U \in \mathcal{U}} \overline{\{x_i \mid i \in U\}}$$

- $\mathbf{Pret}(\mathbf{T}, \mathbf{Set})$

  $$\mathrm{Ult}(\mathbf{Pret}(\mathbf{T}, \mathbf{Set})) \longrightarrow \mathbf{Pret}(\mathbf{T}, \mathbf{Set})$$

  $$(I, \mathcal{U}, (\mathfrak{M}_i)_{i \in I}) \longmapsto \prod_{\mathcal{U}}(\mathfrak{M}_\bullet)$$

- $\mathbf{2} = \boxed{0 \longrightarrow 1}$

  $$\mathrm{Ult}(\mathbf{2}) \longrightarrow \mathbf{2}$$

  $$(I, \mathcal{U}, (c_i)_{i \in I}) \longmapsto n \quad \text{s.t. } [\![c_i = n]\!] \in \mathcal{U}$$

- $$\begin{tikzcd} \mathbf{2} \arrow[r, hook] \arrow[r, phantom, bend left=20, "\perp"] & \mathbf{Set} \arrow[l, bend right=40] \end{tikzcd}$$ is an adjunction of Ult-pseudoalgebras

Basic results

Proposition

For $\mathbf{A}$ an ultracategory the hom-category $\mathrm{Ult}\text{-}\mathbf{PsAlg}(\mathbf{A},\mathbf{Set})$ is a pretopos

Proposition

The 2-functor $\mathrm{Ult}\colon \mathbf{Cat} \longrightarrow \mathbf{Cat}$ takes a Grothendieck fibration to a Grothendieck fibration

Theorem

The ultracompletion $\mathrm{Ult}(\mathbf{C})$ is an ultracategory in the sense of Makkai

Proof

$\hom_{\mathrm{Ult}(\mathbf{C})}\big((K,\mathcal{W},(c_k)_{k\in K}),-\big)\colon \mathrm{Ult}(\mathbf{C}) \longrightarrow \mathbf{Set}$ preserves ultraproducts

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Basic results

Proposition

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Corollary

$\mathrm{Ult}\text{-}\mathbf{PsAlg} \hookrightarrow \mathrm{M}\text{-}\mathbf{Ultcat}$

Corollary

For $\mathbf{P}$ a pretopos with a generating set
the evaluation functor $\mathrm{Ev}\colon \mathbf{P} \longrightarrow \mathrm{Ult}\text{-}\mathbf{PsAlg}(\mathbf{Pret}(\mathbf{P},\mathbf{Set}),\mathbf{Set})$ is an equivalence

Lurie's ultracategories

Definition

For $\mathcal{U}$ an ultrafilter on I and $(\mathcal{V}_i)_{i\in I}$ an I-indexed family of ultrafilters on J their **composition** $\mathcal{U}\cdot\mathcal{V}_\bullet$ consists of those subsets of J such that

$$W \in \mathcal{U}\cdot\mathcal{V}_\bullet \overset{\mathrm{df}}{\Leftrightarrow} \left\{ i\in I \,\middle|\, W\in\mathcal{V}_i \right\} \in \mathcal{U}$$

$$\begin{tikzcd}
 & \mathbf{2}^{I\times J} \arrow[rr, "\sim"] & & \mathbf{2}^{\sum_{i\in I} J} \arrow[dr, "\sum_{\mathcal{U}} \mathcal{V}_\bullet"] & \\
\mathbf{2}^J \arrow[ur, "\mathbf{2}^{\mathrm{p}_2}"] \arrow[rrrr, "\mathrm{ch}\left(\mathcal{U}\cdot\mathcal{V}_\bullet\right)"] \arrow[drr, "\langle \mathrm{ch}\left(\mathcal{V}_i\right)\rangle_{i\in I}"'] & & & & \mathbf{2} \\
 & & \mathbf{2}^I \arrow[urr, "\mathrm{ch}\left(\mathcal{U}\right)"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{2}^I \arrow[r, "\langle \mathrm{ch}\left(i^{\in}\right)\rangle_{i\in I}"] \arrow[rr, bend right, "\mathrm{ch}\left(\mathcal{U}\cdot\left(i^{\in}\right)_{i\in I}\right) = \mathrm{ch}\left(\mathcal{U}\right)"'] & \mathbf{2}^I \arrow[r, "\mathrm{ch}\left(\mathcal{U}\right)"] & \mathbf{2}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{2}^J \arrow[r, "\langle \mathrm{ch}\left(\mathcal{V}_i\right)\rangle_{i\in I}"] \arrow[rr, bend right, "\mathrm{ch}\left(\mathcal{V}_{i_0}\right)"'] & \mathbf{2}^I \arrow[r, "\mathrm{ch}\left(i_0^{\in}\right)"] & \mathbf{2}
\end{tikzcd}$$

Lurie's ultracategories

Definition

An **ultracategory** (in the sense of Lurie) is a category $\mathbf{C}$ endowed with

- a functor $[\mathcal{U}]\colon \mathbf{C}^I \longrightarrow \mathbf{C}$ for each I and ultrafilter $\mathcal{U}$ on I
- natural transformations

$$\begin{tikzcd}
& \mathbf{C}^I \arrow[dr, "{[\mathcal{U}]}"] & \\
\mathbf{C}^J \arrow[ur, "{\langle[\mathcal{V}_i]\rangle_{i\in I}}"] \arrow[rr, "{[\mathcal{U}\cdot\mathcal{V}_\bullet]}"'] \arrow[rr, phantom, shift left=6, "{\Delta_{\mathcal{U},\mathcal{V}_\bullet}\nearrow\cdot}"] & & \mathbf{C}
\end{tikzcd}$$

- natural isomorphisms

$$\begin{tikzcd}
\mathbf{C}^I \arrow[r, shift left=3, "{[i^\in]}"] \arrow[r, shift right=3, "P_i"'] \arrow[r, phantom, "{\varepsilon_{I,i}\downarrow\cdot}"] & \mathbf{C}
\end{tikzcd}$$

for $i \in I$

such that...

Lurie's ultracategories

such that

$$\begin{tikzcd}
\mathbf{C}^J \arrow[r, "\langle[\mathcal{V}_i]\rangle_{i\in I}"] \arrow[rr, bend right=40, "{[\mathcal{V}_{i_0}]}"', ""{name=C}] & \mathbf{C}^I \arrow[r, shift left=2, "{[i_0^{\in}]}", ""{name=A, below}] \arrow[r, shift right=2, "\mathrm{P}_{i_0}"', ""{name=B}] & \mathbf{C} \arrow[from=A, to=B, "\varepsilon_{I,i_0}"', "\cdot"] \arrow[from=B, to=C, phantom, "\parallel"]
\end{tikzcd}$$

is inverse of

$$\begin{tikzcd}
 & \mathbf{C}^I \arrow[dr, "{[i_0^{\in}]}"] & \\
\mathbf{C}^J \arrow[ur, "\langle[\mathcal{V}_i]\rangle_{i\in I}"] \arrow[rr, "{[i_0^{\in}\cdot\mathcal{V}_\bullet] = [\mathcal{V}_{i_0}]}"', ""{name=D}] & & \mathbf{C} \arrow[from=D, to=1-2, shorten >=12pt, "\Delta_{i_0^{\in},\mathcal{V}_\bullet}", "\cdot"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{C}^J \arrow[r, "\langle[\mathcal{V}_i]\rangle_{i\in I}"] \arrow[dr, "{[\mathcal{U}\cdot\mathcal{V}_\bullet]}"{description}, ""{name=M}] & \mathbf{C}^I \arrow[d, "{[\mathcal{U}]}", ""{name=R, left}] \\
\mathbf{C}^L \arrow[u, "\langle[\mathcal{W}_j]\rangle_{j\in J}"] \arrow[r, "{[(\mathcal{U}\cdot\mathcal{V}_\bullet)\cdot\mathcal{W}_*]}"', ""{name=B}] & \mathbf{C} \arrow[from=B, to=M, "\Delta", "\cdot"'] \arrow[from=M, to=R, "\Delta", "\cdot"']
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbf{C}^J \arrow[r, "\langle[\mathcal{V}_i]\rangle_{i\in I}", ""{name=T, below}] & \mathbf{C}^I \arrow[d, "{[\mathcal{U}]}", ""{name=R2, left}] \\
\mathbf{C}^L \arrow[u, "\langle[\mathcal{W}_j]\rangle_{j\in J}"] \arrow[ur, "{\langle[(\mathcal{V}_i\cdot\mathcal{W}_*)]\rangle_{i\in I}}"{description}, ""{name=M2}] \arrow[r, "{[\mathcal{U}\cdot(\mathcal{V}_\bullet\cdot\mathcal{W}_*)]}"', ""{name=B2}] & \mathbf{C} \arrow[from=M2, to=T, "\Delta", "\cdot"'] \arrow[from=B2, to=R2, "\Delta", "\cdot"']
\end{tikzcd}$$

$\Delta_{\mathcal{U},(f(i)^{\in})_{i\in I}} \colon [\mathcal{U}\cdot(f(i)^{\in})_{i\in I}] \overset{\cdot}{\longrightarrow} [\mathcal{U}]\langle[f(i)^{\in}]\rangle_{i\in I}$ is iso when $f \colon I \rightarrowtail J$

Set is an ultracategory in the sense of Lurie

- $$\begin{tikzcd} {[\mathcal{U}]} : \mathbf{Set}^I \arrow[r, "\prod_{\mathcal{U}}"] & \mathbf{Set} \\ S_\bullet = (S_i)_{i\in I} \arrow[r, mapsto] & \operatorname{colim}_{U\in\mathcal{U}} \prod_{i\in U} S_i \end{tikzcd}$$

- $$\begin{tikzcd} & \mathbf{Set}^I \arrow[dr, "{[\mathcal{U}]}"] & \\ \mathbf{Set}^J \arrow[ur, "{\langle[\mathcal{V}_i]\rangle_{i\in I}}"] \arrow[rr, "{[\mathcal{U}\cdot\mathcal{V}_\bullet]}"'] & {} \arrow[u, Rightarrow, "\Delta_{\mathcal{U},\mathcal{V}_\bullet}"] & \mathbf{Set} \end{tikzcd}$$

- $$\begin{tikzcd} \mathbf{Set}^I \arrow[r, bend left, "{[i^\in]}", ""{name=U, below}] \arrow[r, bend right, "\mathrm{P}_i"', ""{name=D}] & \mathbf{Set} \arrow[from=U, to=D, Rightarrow, "\varepsilon_{I,i}"'] \end{tikzcd}$$

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- $$\begin{tikzcd} {[\mathcal{U}]} : \mathbf{Set}^I \arrow[r, "\prod_{\mathcal{U}}"] & \mathbf{Set} \\ S_\bullet = (S_i)_{i\in I} \arrow[r, mapsto] & \operatorname{colim}_{U\in\mathcal{U}} \prod_{i\in U} S_i \end{tikzcd}$$

- $$\begin{tikzcd} & \mathbf{Set}^I \arrow[dr, "\prod_{\mathcal{U}}"] & \\ \mathbf{Set}^J \arrow[ur, "\langle \prod_{\mathcal{V}_i} \rangle_{i\in I}"] \arrow[rr, "\prod_{\mathcal{U}\cdot\mathcal{V}_\bullet}"'] \arrow[rr, phantom, bend left=20, "\Delta_{\mathcal{U},\mathcal{V}_\bullet} \nearrow \cdot"] & & \mathbf{Set} \end{tikzcd}$$

- $$\begin{tikzcd} \mathbf{Set}^I \arrow[r, bend left, "\prod_{j\in}"] \arrow[r, bend right, "P_i"'] \arrow[r, phantom, "\varepsilon_{I,i} \Downarrow \cdot"] & \mathbf{Set} \end{tikzcd}$$

Set is an ultracategory in the sense of Lurie

- $$\begin{tikzcd}[\mathcal{U}] : \mathbf{Set}^I \arrow[r, "\prod_{\mathcal{U}}"] & \mathbf{Set} \\ S_\bullet = (S_i)_{i\in I} \arrow[r, mapsto] & \operatorname{colim}_{U\in\mathcal{U}} \prod_{i\in U} S_i\end{tikzcd}$$

- $$\begin{tikzcd} & \mathbf{Set}^I \arrow[dr, "\prod_{\mathcal{U}}"] & \\ \mathbf{Set}^J \arrow[ur, "\langle \prod_{\mathcal{V}_i} \rangle_{i\in I}"] \arrow[rr, "\prod_{\mathcal{U}\cdot\mathcal{V}_\bullet}"', ""{name=B}] & & \mathbf{Set} \arrow[from=B, to=1-2, shorten >=20pt, "\Delta_{\mathcal{U},\mathcal{V}_\bullet}", "\cdot"'] \end{tikzcd}$$

$$\begin{tikzcd}\prod_{\mathcal{U}\cdot\mathcal{V}_\bullet}(S_*) \arrow[r, tail] \arrow[rr, bend right, "\Delta_{\mathcal{U},\mathcal{V}_\bullet,S_*}"'] & \prod_{\Sigma_{\mathcal{U}}\mathcal{V}_\bullet}(\hat{S}_*) \arrow[r, "\sim"] & \prod_{\mathcal{U}}(\prod_{\mathcal{V}_\bullet}(S_*))\end{tikzcd}$$

- $$\begin{tikzcd}\mathbf{Set}^I \arrow[r, bend left, "\prod_{j\in}", ""{name=T, below}] \arrow[r, bend right, "\mathrm{P}_i"', ""{name=P, above}] & \mathbf{Set} \arrow[from=T, to=P, "\varepsilon_{I,i}"', "\cdot"]\end{tikzcd}$$

$$\begin{tikzcd}\prod_{j\in}(S_\bullet) \arrow[r, "\sim", "\varepsilon_{I,i,S_\bullet}"'] & S_i\end{tikzcd}$$

Set is an ultracategory in the sense of Lurie

- $$\begin{tikzcd}
[\mathcal{U}]:\mathbf{Set}^I \arrow[r, "\prod_{\mathcal{U}}"] & \mathbf{Set} \\
S_\bullet = (S_i)_{i\in I} \arrow[r, mapsto] & \operatorname{colim}_{U\in\mathcal{U}} \prod_{i\in U} S_i
\end{tikzcd}$$

- $$\begin{tikzcd}
 & \mathbf{Set}^I \arrow[dr, "\prod_{\mathcal{U}}"] \arrow[d, phantom, "\Delta_{\mathcal{U},\mathcal{V}_\bullet} \Uparrow \cdot"] & \\
\mathbf{Set}^J \arrow[ur, "\langle\prod_{\mathcal{V}_i}\rangle_{i\in I}"] \arrow[rr, "\prod_{\mathcal{U}\cdot\mathcal{V}_\bullet}"'] & {} & \mathbf{Set}
\end{tikzcd}$$

  $$\begin{tikzcd}
\prod_{\mathcal{U}\cdot\mathcal{V}_\bullet}(S_*) \arrow[r, tail] \arrow[rr, bend right, "\Delta_{\mathcal{U},\mathcal{V}_\bullet,S_*}"'] & \prod_{\Sigma_{\mathcal{U}}\mathcal{V}_\bullet}(\hat{S}_*) \arrow[r, "\sim"] & \prod_{\mathcal{U}}(\prod_{\mathcal{V}_\bullet}(S_*))
\end{tikzcd}$$

- $$\begin{tikzcd}
\mathbf{Set}^I \arrow[r, bend left, "\prod_{i\in}"] \arrow[r, bend right, "P_i"'] \arrow[r, phantom, "\varepsilon_{I,i}\Downarrow\cdot"] & \mathbf{Set}
\end{tikzcd}$$

  $$\prod_{i\in}(S_\bullet) \xrightarrow[\varepsilon_{I,i,S_\bullet}]{\sim} S_i$$

- for $f : I \rightarrowtail J$ $$\prod_{\mathcal{U}\cdot(f(i)^{\in})_{i\in I}}(S_\bullet) \xrightarrow[\Delta_{\mathcal{U},(f(i)^{\in})_{i\in I},S_\bullet}]{\sim} \prod_{\mathcal{U}}(\prod_{f(i)^{\in}}(S_\bullet)_{i\in I})$$

Set is an ultracategory in the sense of Lurie

- $[\mathcal{U}] : \mathbf{Set}^I \xrightarrow{\prod_{\mathcal{U}}} \mathbf{Set}$

  $S_\bullet = (S_i)_{i\in I} \longmapsto \operatorname{colim}_{U\in\mathcal{U}} \prod_{i\in U} S_i$

- $$\begin{tikzcd} & \mathbf{Set}^I \arrow[dr, "\prod_{\mathcal{U}}"] & \\ \mathbf{Set}^J \arrow[ur, "\langle \prod_{\mathcal{V}_i} \rangle_{i\in I}"] \arrow[rr, "\prod_{\mathcal{U}\cdot\mathcal{V}_\bullet}"'] & {} \arrow[u, "\Delta_{\mathcal{U},\mathcal{V}_\bullet}", "\cdot"'] & \mathbf{Set} \end{tikzcd}$$

  $$\begin{tikzcd} \prod_{\mathcal{U}\cdot\mathcal{V}_\bullet}(S_*) \arrow[r, tail] \arrow[rr, bend right, "\Delta_{\mathcal{U},\mathcal{V}_\bullet,S_*}"'] & \prod_{\Sigma_{\mathcal{U}}\mathcal{V}_\bullet}(\hat{S}_*) \arrow[r, "\sim"] & \prod_{\mathcal{U}}(\prod_{\mathcal{V}_\bullet}(S_*)) \end{tikzcd}$$

- $$\begin{tikzcd} \mathbf{Set}^I \arrow[r, bend left, "\prod_{i^\in}"{name=A}] \arrow[r, bend right, "\mathrm{P}_i"'{name=B}] \arrow[from=A, to=B, "\varepsilon_{I,i}"', "\cdot"] & \mathbf{Set} \end{tikzcd}$$

  $$\prod_{i^\in}(S_\bullet) \xrightarrow[\varepsilon_{I,i,S_\bullet}]{\sim} S_i$$

- for $f : I \hookrightarrow J$ $$\prod_{\mathcal{U}\cdot(f(i)^\in)_{i\in I}}(S_\bullet) \xrightarrow[\Delta_{\mathcal{U},(f(i)^\in)_{i\in I},S_\bullet}]{\sim} \prod_{\mathcal{U}}(\prod_{f(i)^\in}(S_\bullet)_{i\in I})$$

Set is an ultracategory in the sense of Lurie

- $$\begin{tikzcd} {[\mathcal{U}]} : \mathbf{Set}^I \arrow[r, "\prod_{\mathcal{U}}"] & \mathbf{Set} \\ S_\bullet = (S_i)_{i\in I} \arrow[r, mapsto] & \operatorname{colim}_{U\in\mathcal{U}} \prod_{i\in U} S_i \end{tikzcd}$$

- $$\begin{tikzcd} & \mathbf{Set}^I \arrow[dr, "\prod_{\mathcal{U}}"] & \\ \mathbf{Set}^J \arrow[ur, "\langle\prod_{\mathcal{V}_i}\rangle_{i\in I}"] \arrow[rr, "\prod_{\mathcal{U}\cdot\mathcal{V}_\bullet}"'] \arrow[from=2-1, to=1-3, phantom, "\Delta_{\mathcal{U},\mathcal{V}_\bullet}\ \overset{\cdot}{\nearrow}"] & & \mathbf{Set} \end{tikzcd}$$
  $$\begin{tikzcd} \prod_{\mathcal{U}\cdot\mathcal{V}_\bullet}(S_*) \arrow[r, tail] \arrow[rr, bend right, "\Delta_{\mathcal{U},\mathcal{V}_\bullet,S_*}"'] & \prod_{\Sigma_{\mathcal{U}}\mathcal{V}_\bullet}(\hat{S}_*) \arrow[r, "\sim"] & \prod_{\mathcal{U}}(\prod_{\mathcal{V}_\bullet}(S_*)) \end{tikzcd}$$

- $$\begin{tikzcd} \mathbf{Set}^I \arrow[r, bend left, "\prod_{i\in}", ""{name=A, below}] \arrow[r, bend right, "P_i"', ""{name=B, above}] \arrow[from=A, to=B, "\epsilon_{I,i}"', "\cdot"] & \mathbf{Set} \end{tikzcd}$$
  $$\begin{tikzcd} \prod_{i\in}(S_\bullet) \arrow[r, "\sim", "\epsilon_{I,i,S_\bullet}"'] & S_i \end{tikzcd}$$

- for $f : I \hookrightarrow J$
  $$\begin{tikzcd} \prod_{\mathcal{U}\cdot(i\in)_{i\in I}}(S_\bullet) \arrow[r, "\sim", "\Delta_{\mathcal{U},(i\in)_{i\in I},S_\bullet}"'] & \prod_{\mathcal{U}}(\prod_{i\in}(S_\bullet)_{i\in I}) \end{tikzcd}$$

Set is an ultracategory in the sense of Lurie

- $$\begin{tikzcd}
{[\mathcal{U}]} : \mathbf{Set}^I \arrow[r, "\prod_{\mathcal{U}}"] & \mathbf{Set} \\
S_\bullet = (S_i)_{i\in I} \arrow[r, mapsto] & \operatorname{colim}_{U\in\mathcal{U}} \prod_{i\in U} S_i
\end{tikzcd}$$

- $$\begin{tikzcd}
 & \mathbf{Set}^I \arrow[dr, "\prod_{\mathcal{U}}"] & \\
\mathbf{Set}^J \arrow[ur, "\langle \prod_{\mathcal{V}_i} \rangle_{i\in I}"] \arrow[rr, "\prod_{\mathcal{U}\cdot\mathcal{V}_\bullet}"', ""{name=D}] & & \mathbf{Set}
\arrow[from=D, to=1-2, shorten <=2pt, shorten >=20pt, "\Delta_{\mathcal{U},\mathcal{V}_\bullet}", "\cdot"']
\end{tikzcd}$$
$$\begin{tikzcd}
\prod_{\mathcal{U}\cdot\mathcal{V}_\bullet}(S_*) \arrow[r, tail] \arrow[rr, bend right, "\Delta_{\mathcal{U},\mathcal{V}_\bullet,S_*}"'] & \prod_{\sum_{\mathcal{U}}\mathcal{V}_\bullet}(\hat{S}_*) \arrow[r, "\sim"] & \prod_{\mathcal{U}}(\prod_{\mathcal{V}_\bullet}(S_*))
\end{tikzcd}$$

- $$\begin{tikzcd}
\mathbf{Set}^I \arrow[r, bend left, "\prod_{i\in}", ""{name=A, below}] \arrow[r, bend right, "\mathrm{P}_i"', ""{name=B}] & \mathbf{Set}
\arrow[from=A, to=B, "\varepsilon_{I,i}"', "\cdot"]
\end{tikzcd}$$
$$\begin{tikzcd}
\prod_{i\in}(S_\bullet) \arrow[r, "\sim", "\varepsilon_{I,i,S_\bullet}"'] & S_i
\end{tikzcd}$$

- for $f : I \hookrightarrow J$
$$\begin{tikzcd}
\prod_{\mathcal{U}\cdot(i\in)_{i\in I}}(S_\bullet) \arrow[r, "\sim", "\Delta_{\mathcal{U},(i\in)_{i\in I},S_\bullet}"'] \arrow[d, equal] & \prod_{\mathcal{U}}(\prod_{i\in}(S_\bullet)_{i\in I}) \arrow[d, "\wr"] \\
\prod_{\mathcal{U}_J}((S_j)_{j\in J}) & \prod_{\mathcal{U}}((S_i)_{i\in I})
\end{tikzcd}$$

Set is an ultracategory in the sense of Lurie

- $[\mathcal{U}] : \mathbf{Set}^I \xrightarrow{\prod_{\mathcal{U}}} \mathbf{Set}$

  $S_\bullet = (S_i)_{i\in I} \longmapsto \operatorname{colim}_{U\in\mathcal{U}} \prod_{i\in U} S_i$

- $$\begin{tikzcd} & \mathbf{Set}^I \arrow[dr, "\prod_{\mathcal{U}}"] & \\ \mathbf{Set}^J \arrow[ur, "\langle \prod_{\mathcal{V}_i} \rangle_{i\in I}"] \arrow[rr, "\prod_{\mathcal{U}\cdot\mathcal{V}_\bullet}"', ""{name=B, above}] & & \mathbf{Set} \arrow[from=B, to=1-2, shorten >=12pt, "\Delta_{\mathcal{U},\mathcal{V}_\bullet}" near end] \end{tikzcd}$$

  $$\begin{tikzcd} \prod_{\mathcal{U}\cdot\mathcal{V}_\bullet}(S_*) \arrow[r, tail] \arrow[rr, bend right, "\Delta_{\mathcal{U},\mathcal{V}_\bullet,S_*}"'] & \prod_{\Sigma_{\mathcal{U}}\mathcal{V}_\bullet}(\hat{S}_*) \arrow[r, "\sim"] & \prod_{\mathcal{U}}(\prod_{\mathcal{V}_\bullet}(S_*)) \end{tikzcd}$$

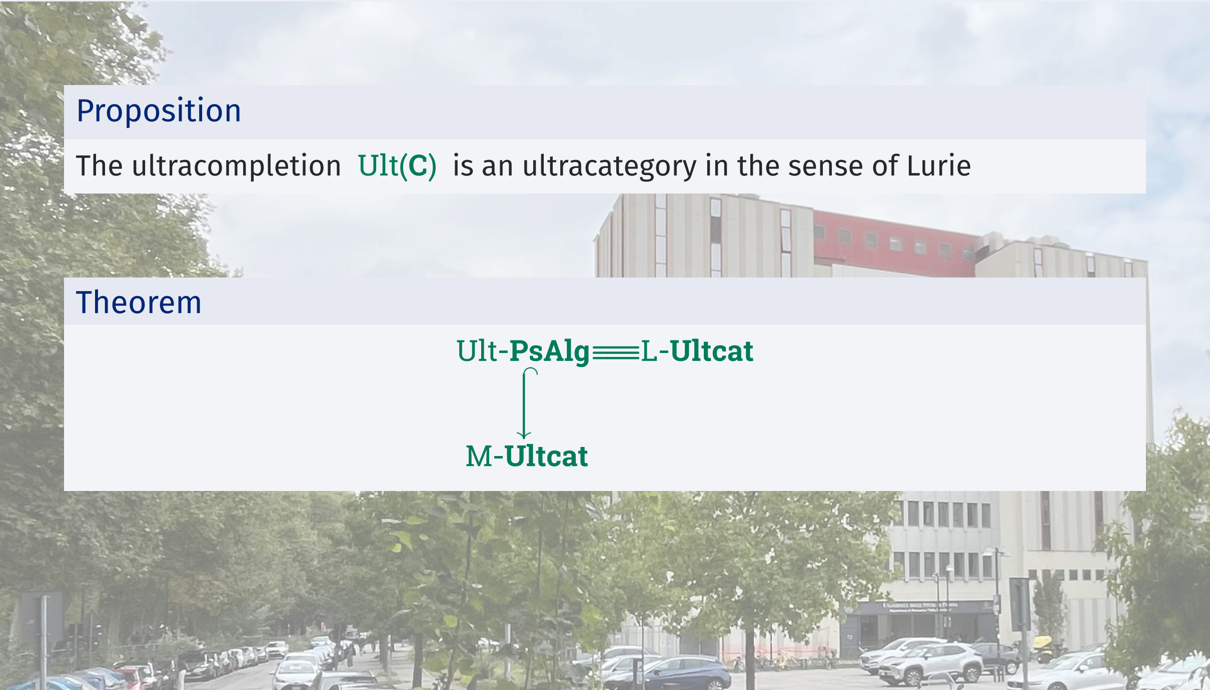
- $$\begin{tikzcd} \mathbf{Set}^I \arrow[r, bend left, "\prod_{i\in}", ""{name=U, below}] \arrow[r, bend right, "P_i"', ""{name=D, above}] & \mathbf{Set} \arrow[from=U, to=D, "\varepsilon_{I,i}"'] \end{tikzcd}$$

  $$\prod_{i\in}(S_\bullet) \xrightarrow[\varepsilon_{I,i,S_\bullet}]{\sim} S_i$$

- for $f : I \hookrightarrow J$

  $$\begin{tikzcd} \prod_{\mathcal{U}\cdot(i\in)_{i\in I}}(S_\bullet) \arrow[r, "\sim", "\Delta_{\mathcal{U},(i\in)_{i\in I},S_\bullet}"'] \arrow[d, equal] & \prod_{\mathcal{U}}(\prod_{i\in}(S_\bullet)_{i\in I}) \arrow[d, "\wr"] \\ \prod_{\mathcal{U}_J}((S_j)_{j\in J}) \arrow[r, "\sim"] & \prod_{\mathcal{U}}((S_i)_{i\in I}) \end{tikzcd}$$

Basic results



Proposition

The ultracompletion $\mathrm{Ult}(\mathbf{C})$ is an ultracategory in the sense of Lurie

Theorem

$$\begin{tikzcd}
\mathrm{Ult}\text{-}\mathbf{PsAlg} \arrow[r, phantom, "\equiv"] \arrow[d, hook] & \mathrm{L}\text{-}\mathbf{Ultcat} \\
\mathrm{M}\text{-}\mathbf{Ultcat} &
\end{tikzcd}$$

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