Università di Genova
DIMA DIPARTIMENTO DI MATEMATICA

Exercises in Completion

Pino Rosolini
with a nod to Jaap van Oosten

Joint work in progress with
Cipriano Junior Cioffo
Jacopo Emmenegger
Fabio Pasquali

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A quick recap on exact completions

Theorem (Carboni–Celia Magno, 1982)

An exact category is an exact completion of a category with finite limits
if and only if

(i) it has enough (regular) projectives

(ii) projectives are closed under finite limits

Theorem (Vitale, 1994; Carboni–Vitale, 1998)

An exact category is an exact completion of a category with weak finite limits
if and only if

(i) it has enough (regular) projectives

In fact, every category with weak finite limits in which every idempotent splits, appears as the full subcategory of the (regular) projectives of its exact completion

A quick recap on exact completions

Theorem (Carboni–Celia Magno, 1982)

Let $\mathbf{C}$ be a category with finite limits
There is a functor $\text{よ}\colon \mathbf{C} \longrightarrow \mathbf{C}_{\mathrm{ex/lex}}$ into an exact category $\mathbf{C}_{\mathrm{ex/lex}}$
The functor preserves finite limits and pre-composition with it gives

an equivalence $\mathbf{EX}(\mathbf{C}_{\mathrm{ex/lex}}, \mathbf{E}) \xlongequal{- \circ \text{よ}} \mathbf{LEX}(\mathbf{C}, \mathbf{E})$ for any exact category $\mathbf{E}$

Theorem (Vitale, 1994)

Let $\mathbf{C}$ be a category with weak finite limits
There is a functor $\text{よ}\colon \mathbf{C} \longrightarrow \mathbf{C}_{\mathrm{ex/wlex}}$ into an exact category $\mathbf{C}_{\mathrm{ex/wlex}}$
The functor is left covering and pre-composition with it gives

an equivalence $\mathbf{EX}(\mathbf{C}_{\mathrm{ex/wlex}}, \mathbf{E}) \xlongequal{- \circ \text{よ}} \mathrm{LftCov}(\mathbf{C}, \mathbf{E})$ for any exact category $\mathbf{E}$

A quick recap on exact completions

A functor $F\colon \mathbf{C} \longrightarrow \mathbf{E}$ is **left covering**

from a category $\mathbf{C}$ with weak finite limits to an exact category $\mathbf{E}$ when

for every weak finite limit

$$\begin{tikzcd}
 & W \arrow[dl] \arrow[dd] \arrow[dr] \arrow[drr] & & \\
D_i \arrow[dr] & & D_j \arrow[dl] & D_k \\
 & D_h & &
\end{tikzcd}$$

the arrow given by universality

$$\begin{tikzcd}
 & FW \arrow[rr, dashed] \arrow[dl] \arrow[dd] \arrow[dr] \arrow[drr] & & \lim FD \arrow[dlll] \arrow[dl] \arrow[d] \arrow[ddll, bend right] \\
FD_i \arrow[dr] & & FD_j \arrow[dl] & FD_k \\
 & FD_h & &
\end{tikzcd}$$

is (regular) epi

E. Vitale. Left Covering Functors. PhD Thesis, Louvain-la-Neuve, 1994

A. Carboni, E. Vitale. Regular and exact completions. *JPAA*, 1998

The exact completion of a category **C** with weak limits

Pseudo-equivalence relations

$$\begin{tikzcd} X_1 \arrow[r, shift left, "\rho'_X"] \arrow[r, shift right, "\rho''_X"'] & X_0 \end{tikzcd}$$: a pair of parallel arrows in **C** such that

there are commutative diagrams

$$\begin{tikzcd}
& & X_0 \\
X_0 \arrow[urr, bend left, "\mathrm{id}_{X_0}"] \arrow[r] \arrow[drr, bend right, "\mathrm{id}_{X_0}"'] & X_1 \arrow[ur, "\rho'_X"'] \arrow[dr, "\rho''_X"] & \\
& & X_0
\end{tikzcd}
\qquad
\begin{tikzcd}
& & X_0 \\
X_1 \arrow[urr, bend left, "\rho''_X"] \arrow[r] \arrow[drr, bend right, "\rho'_X"'] & X_1 \arrow[ur, "\rho'_X"'] \arrow[dr, "\rho''_X"] & \\
& & X_0
\end{tikzcd}
\qquad
\begin{tikzcd}
& & & X_0 \\
& & X_1 \arrow[ur, "\rho'_X"'] \arrow[dr, "\rho''_X"] & \\
X_1 \arrow[uurrr, bend left, "\rho'_X"] \arrow[ddrrr, bend right, "\rho''_X"'] & X_2 \arrow[l] \arrow[ur] \arrow[dr] & \text{weak pullback} & X_0 \\
& & X_1 \arrow[ur, "\rho'_X"'] \arrow[dr, "\rho''_X"] & \\
& & & X_0
\end{tikzcd}$$

The exact completion of a category **C** with weak limits

The category $\mathbf{C}_{\mathrm{ex/wlex}}$

Objects are ps.-equiv.rel.

$$\begin{tikzcd}
X_1 \arrow[d, shift right, "\rho'_X"'] \arrow[d, shift left, "\rho''_X"] & & Y_1 \arrow[d, shift right, "\rho'_Y"'] \arrow[d, shift left, "\rho''_Y"] \\
X_0 \arrow[rr, "{[f]}"'] & & Y_0
\end{tikzcd}$$

An arrow is an equiv.class of fs such that there is

$$\begin{tikzcd}
X_1 \arrow[rr] \arrow[dd, "\rho'_X"'] \arrow[dr, "\rho''_X"] & & Y_1 \arrow[dd, "\rho'_Y"'] \arrow[dr, "\rho''_Y"] & \\
& X_0 \arrow[rr, "f"] & & Y_0 \\
X_0 \arrow[rr, "f"'] & & Y_0 &
\end{tikzcd}$$

$[f] = [g]$ when there is

$$\begin{tikzcd}
& & Y_1 \arrow[dr, "\rho''_Y"] \arrow[ddl, "\rho'_Y"'] & \\
X_0 \arrow[urr] \arrow[rrr, "g"] \arrow[dr, "f"'] & & & Y_0 \\
& Y_0 & &
\end{tikzcd}$$

The exact completion of a category C with weak limits

A. Carboni, R. Celia Magno. The free exact category on a left exact one. *JAMS*, 1982

E. Robinson, G. R. Colimit completions and the effective topos. *JSL*, 1990

P.J. Freyd, A. Scedrov. Categories Allegories. 1991

E. Vitale. Left Covering Functors. PhD Thesis, Louvain-la-Neuve, 1994

A. Carboni. Some free constructions in realizability and proof theory. *JPAA*, 1995

H. Hu. Flat functors and free exact categories. *JAMS*, 1996

H. Hu, W. Tholen. A note on free regular and exact completions and their infinitary generalizations. *TAC*, 1996

A. Carboni, E. Vitale. Regular and exact completions. *JPAA*, 1998

A problem about the 2-functor $(-)_{\mathrm{ex/wlex}}$

Theorem (Carboni–Celia Magno, 1982)

Let $\mathbf{C}$ be a category with finite limits
There is a functor $\text{よ}\colon \mathbf{C} \longrightarrow \mathbf{C}_{\mathrm{ex/lex}}$ into an exact category $\mathbf{C}_{\mathrm{ex/lex}}$
The functor preserves finite limits and pre-composition with it gives

an equivalence $\mathbf{EX}(\mathbf{C}_{\mathrm{ex/lex}}, \mathbf{E}) \overset{-\circ\text{よ}}{\equiv\!\equiv\!\equiv} \mathbf{LEX}(\mathbf{C}, \mathbf{E})$ for any exact category $\mathbf{E}$

Theorem (Vitale, 1994)

Let $\mathbf{C}$ be a category with weak finite limits
There is a functor $\text{よ}\colon \mathbf{C} \longrightarrow \mathbf{C}_{\mathrm{ex/wlex}}$ into an exact category $\mathbf{C}_{\mathrm{ex/wlex}}$
The functor is left covering and pre-composition with it gives

an equivalence $\mathbf{EX}(\mathbf{C}_{\mathrm{ex/wlex}}, \mathbf{E}) \overset{-\circ\text{よ}}{\equiv\!\equiv\!\equiv} \mathrm{LftCov}(\mathbf{C}, \mathbf{E})$ for any exact category $\mathbf{E}$

A problem about the 2-functor $(-)_{\mathrm{ex/wlex}}$

From Carboni and Vitale's *Regular and exact completions*

As a left covering functor defined on a left exact category is exactly a left exact functor, we have, as a particular case of Theorem 29, the main theorem contained in (Carboni and Celia Magno, 1982); as it is shown there, the universal property of the exact completion of a left exact category becomes part of the left biadjoint to the obvious forgetful 2-functor $\mathbf{EX} \longrightarrow \mathbf{LEX}$, where $\mathbf{EX}$ is the 2-category of exact categories and exact functors, and $\mathbf{LEX}$ is the 2-category of left exact categories and left exact functors

The question naturally arising is then whether, with a good choice of morphisms between weakly lex categories, the universal property stated in Theorem 29 becomes part of the analogous biadjunction between exact categories and weakly lex ones

The answer is ***negative***

E. Vitale. Left Covering Functors. PhD Thesis, Louvain-la-Neuve, 1994

A. Carboni, E. Vitale. Regular and exact completions. *JPAA*, 1998

Doctrines, primary, elementary, and existential

A functor $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is a **doctrine** if $\mathbf{C}$ has finite products

A doctrine $P^{\mathrm{op}}\colon \mathbf{C} \longrightarrow \mathbf{Pos}$ is

primary if the fibres have natural finite products $P(C) \times P(C) \xrightarrow{\wedge} P(C)$

elementary if the reindexings along diagonals have a natural left adjoint

$$\begin{tikzcd}
P(C \times D \times D) \arrow[r, bend left, "P\left(C \times D \xrightarrow{\mathrm{id}_C \times \Delta_D} C \times D \times D\right)"] \arrow[r, phantom, "\top"] & P(C \times D) \arrow[l, bend left]
\end{tikzcd}$$

and satisfy FR

existential if the reindexings along terminal arrows have a natural left adjoint

$$\begin{tikzcd}
P(C \times 1) \arrow[r, bend left, "P\left(C \times D \xrightarrow{\mathrm{id}_C \times !_D} C \times 1\right)"] \arrow[r, phantom, "\top"] & P(C \times D) \arrow[l, bend left]
\end{tikzcd}$$

and satisfy FR

F.W. Lawvere. Adjointness in foundations. *Dialectica*, 1969

F.W. Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. **Applications of Categorical Algebra**, 1970

A.M. Pitts. Categorical Logic. **Handbook of Logic in Computer Science**, 2000

Doctrines, primary, elementary, and existential

Examples

For $\mathbf{C}$ a category with finite limits
the **subobject** functor $\mathrm{Sub}_{\mathbf{C}}\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is elementary
The case $\mathbf{C} := \mathbf{Set}$ is the standard contravariant powerset functor

For $\mathbf{R}$ a regular category
the functor $\mathrm{Sub}_{\mathbf{R}}\colon \mathbf{R}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is elementary existential

For S an inf-semilattice
the functor $\mathsf{S}^{(-)}\colon \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is primary
$\mathsf{S}^{(-)}\colon \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is elementary if and only if C has a least element (and δ_C is "Kroneker delta")
$\mathsf{S}^{(-)}\colon \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is existential if and only is S is a frame

For $\mathcal{T}$ a theory in the first order language $\mathcal{L}$
the **Lindenbaum–Tarski** functor $\mathrm{LT}_{\mathcal{T}}\colon \mathbf{Ctx}_{\mathcal{L}}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is elementary existential

Doctrines, primary, elementary, and existential

Characterization of doctrines as orders in the topos $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$

A doctrine $P^{\mathrm{op}} \colon \mathbf{C} \longrightarrow \mathbf{Pos}$ is precisely an object of $\mathrm{Pos}([\mathbf{C}^{\mathrm{op}}, \mathbf{Set}])$

primary if and only if the po-object $\mathbf{P}$ on the presheaf

$\mathbf{C}^{\mathrm{op}} \xrightarrow{P} \mathbf{Pos} \xrightarrow{|-|} \mathbf{Set}$ is an inf-semilattice

So $\mathbf{P}^2 \xrightarrow{\mathrm{cod}} \mathbf{P}$ is a Grothendieck fibration in $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$

existential if and only if each diagonal functor $\Delta \colon \mathbf{P} \longrightarrow \mathbf{P}^A$ has a **fibered** left adjoint $\bigvee \colon \mathbf{P}^A \longrightarrow \mathbf{P}$

$$\forall_{x:|P|^A} \forall_{p,q:|P|} \left[(p \wedge \textstyle\bigvee_{a:A} x_a) \leq q \Leftrightarrow (p \wedge x_a) \leq q \right]$$

elementary if and only if the functor $[\![- = 1]\!] \colon \mathbf{P} \longrightarrow \Omega$ has a **fibered** left adjoint $\omega \colon \Omega \longrightarrow \mathbf{P}$

$$\forall_{w:\Omega} \forall_{p,q:|P|} \left[(p \wedge \omega(w)) \leq q \Leftrightarrow (w \Rightarrow [\![(p \wedge q) = p]\!]) \right]$$

Basic notations

Since $C \times 1 \xrightarrow{\sim} C$

- for an existential doctrine there are also left adjoints

$$\begin{tikzcd}
P(C) \arrow[rd, shift left=2, "P(\mathrm{p}_1)"] \arrow[rd, phantom, "\top"] & \\
P(C \times 1) \arrow[u, "\wr"] \arrow[r, shift left] \arrow[r, phantom, "\top"] & P(C \times D) \arrow[l, shift left] \arrow[lu, shift left=2, "\exists_{\mathrm{p}_1}"]
\end{tikzcd}$$

- for an elementary doctrine the left adjoints are determined from specific objects δ_C in the fibre $P(C \times C)$ such that

1. $P(\mathrm{p}_1)(\beta) \wedge \delta_C \leq P(\mathrm{p}_2)(\beta)$ for every β in $P(C)$
2. $\top_C \leq P(\Delta_C)(\delta_C)$
3. $P(\langle \mathrm{p}_1, \mathrm{p}_3 \rangle)(\delta_C) \wedge P(\langle \mathrm{p}_2, \mathrm{p}_4 \rangle)(\delta_{C'}) \leq \delta_{C \times C'}$

Basic notations

Since $C\times 1 \xrightarrow{\sim} C$

- for an existential doctrine there are also left adjoints

$$\begin{tikzcd}
P(C) \arrow[dr, shift left=2, "P(\mathrm{p}_1)"] \arrow[dr, phantom, "\top"] & \\
P(C\times 1) \arrow[u, "\wr"] \arrow[r, shift left=2] \arrow[r, phantom, "\top"] & P(C\times D) \arrow[l, shift left=2] \arrow[ul, shift left=2, "\exists_{\mathrm{p}_1}"]
\end{tikzcd}$$

- for an elementary doctrine the left adjoints are determined from specific objects δ_C in the fibre $P(C\times C)$ such that

1. $P(\mathrm{p}_1)(\beta)\wedge\delta_C \leq P(\mathrm{p}_2)(\beta)$ for every β in $P(C)$ $\quad\beta(x)\wedge x=y \vdash \beta(y)$
2. $\top_C \leq P(\Delta_C)(\delta_C)$ $\quad\top \vdash x=x$
3. $P(\langle\mathrm{p}_1,\mathrm{p}_3\rangle)(\delta_C)\wedge P(\langle\mathrm{p}_2,\mathrm{p}_4\rangle)(\delta_{C'})\leq\delta_{C\times C'}$ $\quad x=y\wedge x'=y' \vdash (x,x')=(y,y')$

Basic notations

Since $C \times 1 \xrightarrow{\sim} C$

- for an existential doctrine there are also left adjoints

$$\begin{tikzcd}
P(C) \arrow[dr, shift left=2, "\mathrm{p}_1^*"] \arrow[dr, phantom, "\top"] & \\
P(C \times 1) \arrow[u, "\sim"] \arrow[r, shift left] \arrow[r, phantom, "\top"] & P(C \times D) \arrow[l, shift left] \arrow[ul, "\exists_{\mathrm{p}_1}"]
\end{tikzcd}$$

- for an elementary doctrine the left adjoints are determined from specific objects δ_C in the fibre $P(C \times C)$ such that

1. $\mathrm{p}_1^*(\beta) \wedge \delta_C \leq \mathrm{p}_2^*(\beta)$ for every β in $P(C)$ $\qquad \beta(x) \wedge x = y \vdash \beta(y)$
2. $\top_C \leq \Delta_C^*(\delta_C)$ $\qquad \top \vdash x = x$
3. $\langle \mathrm{p}_1, \mathrm{p}_3 \rangle^*(\delta_C) \wedge \langle \mathrm{p}_2, \mathrm{p}_4 \rangle^*(\delta_{C'}) \leq \delta_{C \times C'}$ $\qquad x = y \wedge x' = y' \vdash (x, x') = (y, y')$

The doctrinal view on exact completions

The doctrine of variations

$\mathbf{C}$: a category with finite products and weak equalisers

The functor $\mathrm{Vrn}_{\mathbf{C}}\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is an elementary existential doctrine

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[dd] & (\mathbf{C}/X)_{\mathrm{po}} \arrow[dd] \\
f \arrow[r, mapsto] & f^* \\
X' \arrow[r, mapsto] & (\mathbf{C}/X')_{\mathrm{po}}
\end{tikzcd}$$

F.W. Lawvere. Adjoints in and among bicategories. **Logic and algebra**, 1996

M. Grandis. Weak subobjects and the epi-monic completion of a category. *JPAA*, 2000

The doctrinal view on exact completions

The doctrine of **variations**

$\mathbf{C}$: a category with finite products and weak equalisers

The functor $\mathrm{Vrn}_{\mathbf{C}} \colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is an elementary existential doctrine

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"{name=F}] & (\mathbf{C}/X)_{\mathrm{po}} \arrow[d, shift right=2, "f^*"'{name=G}] \\
X' \arrow[r, mapsto] & (\mathbf{C}/X')_{\mathrm{po}} \arrow[u, shift right=2, bend right, "f \circ - = \exists_f"'] \arrow[from=F, to=G, mapsto, shorten=2ex] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

F.W. Lawvere. Adjoints in and among bicategories. **Logic and algebra**, 1996

M. Grandis. Weak subobjects and the epi-monic completion of a category. *JPAA*, 2000

The doctrinal view on exact completions

Pseudo-equivalence relations are “equivalence relations”

$$\begin{tikzcd}
X_1 \arrow[r, shift left, "\rho'_X"] \arrow[r, shift right, "\rho''_X"'] & X_0
\end{tikzcd}$$

$$\rho_X := \langle \rho'_X, \rho''_X \rangle \in \left(\mathbf{C}/X_0 \times X_0\right)_{\mathrm{po}}$$

$$\begin{tikzcd}
 & & X_0 \\
X_0 \arrow[r] \arrow[urr, bend left, "\mathrm{id}_{X_0}"] \arrow[drr, bend right, "\mathrm{id}_{X_0}"'] & X_1 \arrow[ur, "\rho'_X"'] \arrow[dr, "\rho''_X"] & \\
 & & X_0
\end{tikzcd}$$

$$x', x'' : X_0 \mid x' = x'' \vdash \rho_X(x', x'')$$

$$x : X_0 \mid \top \vdash \rho_X(x, x)$$

$$\begin{tikzcd}
 & & X_0 \\
X_1 \arrow[r] \arrow[urr, bend left, "\rho''_X"] \arrow[drr, bend right, "\rho'_X"'] & X_1 \arrow[ur, "\rho'_X"'] \arrow[dr, "\rho''_X"] & \\
 & & X_0
\end{tikzcd}$$

$$x', x'' : X_0 \mid \rho_X(x', x'') \vdash \rho_X(x'', x')$$

The doctrinal view on exact completions

Pseudo-equivalence relations are “equivalence relations”

$$\begin{tikzcd}
X_1 \arrow[r, shift left, "\rho'_X"] \arrow[r, shift right, "\rho''_X"'] & X_0
\end{tikzcd}$$

$$\rho_X := \langle \rho'_X, \rho''_X \rangle \in \left(\mathbf{C}/X_0 \times X_0\right)_{\mathrm{po}}$$

$$\begin{tikzcd}
& & & X_0 \\
& & X_1 \arrow[ur, "\rho'_X"'] \arrow[dr, "\rho''_X"] & \\
X_1 \arrow[uurrr, bend left, "\rho'_X"] \arrow[ddrrr, bend right, "\rho''_X"'] & X_2 \arrow[l] \arrow[ur] \arrow[dr] & \text{weak pullback} & X_0 \\
& & X_1 \arrow[ur, "\rho'_X"'] \arrow[dr, "\rho''_X"] & \\
& & & X_0
\end{tikzcd}$$

$$x', x'', x''' : X_0 \mid \rho_X(x', x'') \wedge \rho_X(x'', x''') \vdash \rho_X(x', x''')$$

The doctrinal view on exact completions

An arrow in $\mathbf{C}_{\mathrm{ex/wlex}}$ is the “graph” of an arrow in $\mathbf{C}$

$$\begin{tikzcd}
& X_1 \arrow[rrr] \arrow[ddl, "\rho'_X"'] \arrow[dr, "\rho''_X"] & & & Y_1 \arrow[ddl, "\rho'_Y"'] \arrow[dr, "\rho''_Y"] & \\
& & X_0 \arrow[rrr, "f"] & & & Y_0 \\
X_0 \arrow[rrr, "f"'] & & & Y_0 & &
\end{tikzcd}$$

$$x', x'' : X_0 \mid \rho_X(x', x'') \vdash \rho_Y(f(x'), f(x''))$$

$[f] = [g]$ if and only if

$$\begin{tikzcd}
& & Y_1 \arrow[dl, "\rho'_Y"'] \arrow[dr, "\rho''_Y"] & \\
X_0 \arrow[urr] \arrow[rrr, "g"] \arrow[dr, "f"'] & & & Y_0 \\
& Y_0 & &
\end{tikzcd}$$

$$x : X_0 \mid \top \vdash \rho_Y(f(x), g(x))$$

The doctrinal view on exact completions

An arrow in $\mathbf{C}_{\mathrm{ex/wlex}}$ is the “graph” of an arrow in $\mathbf{C}$

$$\begin{tikzcd}
& X_1 \arrow[rrr] \arrow[ddl, "\rho'_X"'] \arrow[dr, "\rho''_X"] & & & Y_1 \arrow[ddl, "\rho'_Y"'] \arrow[dr, "\rho''_Y"] & \\
& & X_0 \arrow[rrr, "f"] & & & Y_0 \\
X_0 \arrow[rrr, "f"'] & & & Y_0 & &
\end{tikzcd}$$

$$x', x'' : X_0 \mid \rho_X(x', x'') \vdash \rho_Y(f(x'), f(x''))$$

$[f] = [g]$ if and only if

$$\begin{tikzcd}
& & Y_1 \arrow[dl, "\rho'_Y"'] \arrow[dr, "\rho''_Y"] & \\
X_0 \arrow[urr] \arrow[rrr, "g"] \arrow[dr, "f"'] & & & Y_0 \\
& Y_0 & &
\end{tikzcd}$$

$$x : X_0 \mid \top \vdash \rho_Y(f(x), g(x))$$

$$x : X_0, y : Y_0 \mid \rho_Y(f(x), y) \dashv\vdash \rho_Y(g(x), y)$$

The doctrinal view on exact completions

The doctrine $\mathrm{Vrn}_{\mathbf{C}}\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ satisfies the **Rule of Choice**

$$\frac{c : C \mid \top \vdash \exists_{d:D}\, \theta(c,d)}{c : C \mid \top \vdash \theta(c, f(c)) \quad \text{some } f\colon C \longrightarrow D}$$

M.E. Maietti, F. Pasquali, G. R. Triposes, exact completions, and Hilbert's ε-operator.
TMJ, 2017

The doctrinal view on exact completions

The doctrine $\mathtt{Vrn}_{\mathbf{C}}\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ satisfies the **Rule of Choice**

Take $[\theta]$ in $\mathtt{Vrn}_{\mathbf{C}}(C \times D)$

$$\begin{tikzcd}
C \arrow[rr] \arrow[ddr, "\top = \mathrm{id}_C"'] & & A \arrow[d, "\theta"] \arrow[ddl, "\exists_{\mathrm{p}_1}\theta"'] \\
& & C \times D \arrow[dl, "\mathrm{p}_1"] \\
& C &
\end{tikzcd}$$

$$\frac{c : C \mid \top \vdash \exists_{d:D}\,\theta(c,d)}{c : C \mid \top \vdash \theta(c, f(c)) \quad \text{some } f\colon C \longrightarrow D}$$

M.E. Maietti, F. Pasquali, G. R. Triposes, exact completions, and Hilbert's ε-operator. *TMJ*, 2017

The doctrinal view on exact completions

The doctrine $\mathrm{Vrn}_{\mathbf{C}}\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ satisfies the **Rule of Choice**

Take $[\theta]$ in $\mathrm{Vrn}_{\mathbf{C}}(C \times D)$

$$\begin{tikzcd}
C \arrow[rr] \arrow[rdd, "\top = \mathrm{id}_C"'] \arrow[rrrdd, bend left, "f"] & & A \arrow[d, "\theta"] \arrow[ldd, "\exists_{\mathrm{p}_1}\theta"'] & \\
& & C \times D \arrow[ld, "\mathrm{p}_1"] \arrow[rd, "\mathrm{p}_2"'] & \\
& C & & D
\end{tikzcd}$$

$$\frac{c : C \mid \top \vdash \exists_{d:D}\, \theta(c, d)}{c : C \mid \top \vdash \theta(c, f(c)) \quad \text{some } f\colon C \longrightarrow D}$$

M.E. Maietti, F. Pasquali, G. R. Triposes, exact completions, and Hilbert's ε-operator. *TMJ*, 2017

The doctrinal view on exact completions

Hence an arrow in $\mathbf{C}_{\mathrm{ex/wlex}}$ is the same as a **map**

$$\begin{tikzcd}
X_1 \arrow[d, shift right, "\rho'_X"'] \arrow[d, shift left, "\rho''_X"] & Y_1 \arrow[d, shift right, "\rho'_Y"'] \arrow[d, shift left, "\rho''_Y"] \\
X_0 \arrow[r, "{[f]}"'] & Y_0
\end{tikzcd}$$

The doctrinal view on exact completions

Hence an arrow in $\mathbf{C}_{\mathrm{ex/wlex}}$ is the same as a **map**

$$\varphi(x,y) := \rho_Y(f(x),y)$$

$$x',x'' : X_0, y',y'' : Y_0 \mid \rho_X(x'',x') \wedge \varphi(x',y') \wedge \rho_Y(y',y'') \vdash \varphi(x'',y'')$$

$$\begin{tikzcd} X_1 \arrow[d, shift right, "\rho'_X"'] \arrow[d, shift left, "\rho''_X"] & Y_1 \arrow[d, shift right, "\rho'_Y"'] \arrow[d, shift left, "\rho''_Y"] \\ X_0 \arrow[r, "{[f]}"'] & Y_0 \end{tikzcd}$$

$$x : X_0, y',y'' : Y_0 \mid \varphi(x,y') \wedge \varphi(x,y'') \vdash \rho_Y(y',y'')$$

$$x : X_0 \mid \top \vdash \exists_{y:Y_0} \varphi(x,y)$$

The doctrinal view on exact completions

Another presentation of $\mathbf{C}_{\mathrm{ex/wlex}}$ when $\mathbf{C}$ has finite products

Objects: $X = (X_0, \rho_X)$ where X_0 is an object in $\mathbf{C}$

ρ_X is an object in the fibre $\mathrm{Vrn}_{\mathbf{C}}(X_0 \times X_0)$ such that

$$x : X_0 \mid \top \vdash \rho_X(x, x)$$
$$x', x'' : X_0 \mid \rho_X(x', x'') \vdash \rho_X(x'', x')$$
$$x', x'', x''' : X_0 \mid \rho_X(x', x'') \wedge \rho_X(x'', x''') \vdash \rho_X(x', x''')$$

Arrows: $\varphi : X \longrightarrow Y$ where φ is an object in the fibre $\mathrm{Vrn}_{\mathbf{C}}(X_0 \times Y_0)$ such that

$$x', x'' : X_0, y', y'' : Y_0 \mid \rho_X(x'', x') \wedge \varphi(x', y') \wedge \rho_Y(y', y'') \vdash \varphi(x'', y'')$$
$$x : X_0, y', y'' : Y_0 \mid \varphi(x, y') \wedge \varphi(x, y'') \vdash \rho_Y(y', y'')$$
$$x : X_0 \mid \top \vdash \exists_{y:Y_0}\, \varphi(x, y)$$

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The doctrinal view on exact completions

The doctrine $\mathrm{Vrn}_{\mathsf{C}}\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ satisfies weak full comprehension

$$\{\!|\varphi|\!\}\colon A \longrightarrow C$$

$$a : A \mid \top \vdash \varphi(\{\!|\varphi|\!\}(a))$$

$$\underline{d : D \mid \top \vdash \varphi(f(d))}$$

$$\begin{tikzcd}
D \arrow[r] \arrow[dr, "f"'] & A \arrow[d, "\{\!|\varphi|\!\}"] \\
& C
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "\{\!|\varphi|\!\}"'] & & A' \arrow[dl, "\{\!|\varphi'|\!\}"] \\
& C &
\end{tikzcd}$$

$$\overline{c : C \mid \varphi(c) \vdash \varphi'(c)}$$

M.E. Maietti, G. R. Quotient completion for the foundation of constructive mathematics. *LU*, 2013

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$$\dfrac{\begin{tikzcd} A \arrow[rr] \arrow[dr, "\{\!|\varphi|\!\}"'] & & A' \arrow[dl, "\{\!|\varphi'|\!\}"] \\ & C & \end{tikzcd}}{c : C \mid \varphi(c) \vdash \varphi'(c)}$$

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$\{\!|\varphi|\!\} := \varphi$

$$\begin{tikzcd}
A \arrow[dr, "\top = \mathrm{id}_A"'] & W \arrow[r] \arrow[d, "\text{w.pb.}", "\{\!|\varphi|\!\}^*(\varphi)"'] & A \arrow[d, "\varphi"] \\
& A \arrow[r, "\{\!|\varphi|\!\}"'] & C
\end{tikzcd}$$

$\{\!|\varphi|\!\}\colon A \longrightarrow C$

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\end{tikzcd}$$

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& C &
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M.E. Maietti, G. R. Quotient completion for the foundation of constructive mathematics. *LU*, 2013

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For $[\varphi]$ in $\mathrm{Vrn}_{\mathbf{C}}(C)$ take

$\{\!|\varphi|\!\} := \varphi$

$$\begin{tikzcd}
A \arrow[rr, bend left] \arrow[dr, "\top = \mathrm{id}_A"'] & W \arrow[r] \arrow[d, "\{\!|\varphi|\!\}^*(\varphi)"] \arrow[dr, phantom, "\text{w.pb.}" very near start] & A \arrow[d, "\varphi"] \\
 & A \arrow[r, "\{\!|\varphi|\!\}"'] & C
\end{tikzcd}$$

$$\{\!|\varphi|\!\} \colon A \longrightarrow C$$

$$a : A \mid \top \vdash \varphi(\{\!|\varphi|\!\}(a))$$

$$\begin{array}{c} d : D \mid \top \vdash \varphi(f(d)) \\ \hline \end{array}$$

$$\begin{tikzcd}
D \arrow[r] \arrow[dr, "f"'] & A \arrow[d, "\{\!|\varphi|\!\}"] \\
 & C
\end{tikzcd}$$

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 & A \arrow[r, "\{\!|\varphi|\!\}"'] & C
\end{tikzcd}$$

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$$\frac{d : D \mid \top \vdash \varphi(f(d))}{\begin{tikzcd}
D \arrow[r] \arrow[dr, "f"'] & A \arrow[d, "\{\!|\varphi|\!\}"] \\
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 & C &
\end{tikzcd}}{c : C \mid \varphi(c) \vdash \varphi'(c)}$$

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& A \arrow[r, "\{\!|\varphi|\!\}"'] & C
\end{tikzcd}$$

$$\begin{tikzcd}
D \arrow[r] \arrow[dr, "\top = \mathrm{id}_D"'] & V \arrow[r] \arrow[d, "f^*(\varphi)"] \arrow[dr, phantom, "\text{w.pb.}"] & A \arrow[d, "\varphi"] \\
& D \arrow[r, "f"'] & C
\end{tikzcd}$$

$$\{\!|\varphi|\!\}\colon A \longrightarrow C$$

$$a : A \mid \top \vdash \varphi(\{\!|\varphi|\!\}(a))$$

$$\begin{array}{c} d : D \mid \top \vdash \varphi(f(d)) \\ \hline \begin{tikzcd} D \arrow[r] \arrow[dr, "f"'] & A \arrow[d, "\{\!|\varphi|\!\}"] \\ & C \end{tikzcd} \end{array}$$

$$\begin{array}{c} \begin{tikzcd} A \arrow[rr] \arrow[dr, "\{\!|\varphi|\!\}"'] & & A' \arrow[dl, "\{\!|\varphi'|\!\}"] \\ & C & \end{tikzcd} \\ \hline c : C \mid \varphi(c) \vdash \varphi'(c) \end{array}$$

M.E. Maietti, G. R. Quotient completion for the foundation of constructive mathematics. *LU*, 2013

The doctrinal view on exact completions

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For $[\varphi]$ in $\mathrm{Vrn}_{\mathbf{C}}(C)$ take

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$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left] \arrow[dr, "\top = \mathrm{id}_A"'] & W \arrow[r] \arrow[d, "\{\!|\varphi|\!\}^*(\varphi)"] \arrow[dr, phantom, "\text{w.pb.}"] & A \arrow[d, "\varphi"] \\
& A \arrow[r, "\{\!|\varphi|\!\}"'] & C
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D \arrow[r] \arrow[rr, bend left] \arrow[dr, "\top = \mathrm{id}_D"'] & V \arrow[r] \arrow[d, "f^*(\varphi)"] \arrow[dr, phantom, "\text{w.pb.}"] & A \arrow[d, "\varphi"] \\
& D \arrow[r, "f"'] & C
\end{tikzcd}$$

$$\dfrac{d : D \mid \top \vdash \varphi(f(d))}{\begin{tikzcd} D \arrow[r] \arrow[dr, "f"'] & A \arrow[d, "\{\!|\varphi|\!\}"] \\ & C \end{tikzcd}}$$

$$\dfrac{\begin{tikzcd} A \arrow[rr] \arrow[dr, "\{\!|\varphi|\!\}"'] & & A' \arrow[dl, "\{\!|\varphi'|\!\}"] \\ & C & \end{tikzcd}}{c : C \mid \varphi(c) \vdash \varphi'(c)}$$

M.E. Maietti, G. R. Quotient completion for the foundation of constructive mathematics. *LU*, 2013

The doctrinal view on exact completions

Yet another presentation of $\mathbf{C}_{\mathrm{ex/wlex}}$ when $\mathbf{C}$ has finite products

Objects: $X = (|X|, \rho_X)$ such that $|X|$ is an object in $\mathbf{C}$

ρ_X is an object in the fibre $P(|X| \times |X|)$ and

$$x', x'' : |X| \,|\, \rho_X(x', x'') \vdash \rho_X(x'', x')$$

$$x', x'', x''' : |X| \,|\, \rho_X(x', x'') \wedge \rho_X(x'', x''') \vdash \rho_X(x', x''')$$

Arrows: $\varphi \colon X \longrightarrow Y$ such that φ is an object in the fibre $P(|X| \times |Y|)$ and

$$x : |X|, y : |Y| \,|\, \varphi(x, y) \vdash \rho_X(x, x) \wedge \rho_Y(y, y)$$

$$x', x'' : |X|, y', y'' : |Y| \,|\, \rho_X(x'', x') \wedge \varphi(x', y') \wedge \rho_Y(y', y'') \vdash \varphi(x'', y'')$$

$$x : |X|, y', y'' : |Y| \,|\, \varphi(x, y') \wedge \varphi(x, y'') \vdash \rho_Y(y', y'')$$

$$x : |X| \,|\, \rho_X(x, x) \vdash \exists_{y:|Y|}\, \varphi(x, y)$$

The exact completion of an e.e.d. $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$

The exact category $P_{\mathrm{ex/eed}}$ —aka the “tripos-to-topos construction”

Objects: $X = (|X|, \rho_X)$ such that $|X|$ is an object in $\mathbf{C}$

ρ_X is an object in the fibre $P(|X| \times |X|)$ and

$$x', x'' : |X| \,|\, \rho_X(x', x'') \vdash \rho_X(x'', x')$$

$$x', x'', x''' : |X| \,|\, \rho_X(x', x'') \wedge \rho_X(x'', x''') \vdash \rho_X(x', x''')$$

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$$x : |X|, y : |Y| \,|\, \varphi(x, y) \vdash \rho_X(x, x) \wedge \rho_Y(y, y)$$

$$x', x'' : |X|, y', y'' : |Y| \,|\, \rho_X(x'', x') \wedge \varphi(x', y') \wedge \rho_Y(y', y'') \vdash \varphi(x', y')$$

$$x : |X|, y', y'' : |Y| \,|\, \varphi(x, y') \wedge \varphi(x, y'') \vdash \rho_Y(y', y'')$$

$$x : |X| \,|\, \rho_X(x, x) \vdash \exists_{y:|Y|}\, \varphi(x, y)$$

D. Higgs. A category approach to Boolean valued models. *Preprint*, 1973

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M. Shulman. Exact completions and small sheaves. *TAC*, 2012

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M.E. Maietti, G. R. Unifying exact completions. *ACS*, 2013

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The exact completion of an e.e.d. $P\colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$

Theorem

$$\begin{tikzcd}
\mathbf{ED} \arrow[r, shift right=2, "(-)_{\mathrm{E}}"'] \arrow[r, phantom, "\top"] \arrow[d, shift right=2, "(\int -)_{\mathrm{x}}"'] \arrow[d, phantom, "\dashv"] & \mathbf{EED} \arrow[l, shift right=2, "U"'] \arrow[d, shift right=2, "(-)_{\mathrm{ex/eed}}"'] \arrow[d, phantom, "\dashv"] \\
\mathbf{LEX} \arrow[u, shift right=2, "\mathrm{Sub}"'] \arrow[r, shift right=2, "(-)_{\mathrm{ex/lex}}"'] \arrow[r, phantom, "\top"] & \mathbf{EX} \arrow[l, shift right=2, "U"'] \arrow[u, shift right=2, "\mathrm{Sub}"']
\end{tikzcd}$$

D. Trotta. The existential completion. *TAC*, 2020

A diagram of the talk

$$\begin{tikzcd}
\mathbf{ED} \arrow[r, shift right=1.5ex, "(-)_{\mathrm{E}}"'] \arrow[r, phantom, "\top"] \arrow[d, shift right=1.5ex, "(\int -)_{\mathrm{x}}"'] \arrow[d, phantom, "\dashv"] & \mathbf{EED} \arrow[l, shift right=1.5ex, "U"'] \arrow[d, shift right=1.5ex, "(-)_{\mathrm{ex/eed}}"'] \arrow[d, phantom, "\dashv"] & & \\
 & & & \mathbf{PrWEq} \arrow[llu, "\mathrm{Vrn}"'] \arrow[lld, "(-)_{\mathrm{ex/wlex}}"] \arrow[ld, hook] \\
\mathbf{LEX} \arrow[u, shift right=1.5ex, "\mathrm{Sub}"'] \arrow[r, shift right=1.5ex, "(-)_{\mathrm{ex/lex}}"'] \arrow[r, phantom, "\top"] & \mathbf{EX} \arrow[u, shift right=1.5ex, "\mathrm{Sub}"'] \arrow[l, shift right=1.5ex, "U"'] & \mathbf{WLEX} \arrow[l, "(-)_{\mathrm{ex/wlex}}"'] &
\end{tikzcd}$$

A diagram of the talk

$$\begin{tikzcd}
\mathbf{ED} \arrow[r, shift right, "(-)_{\mathrm{E}}"'] \arrow[d, shift right, "(\int -)_{\mathrm{x}}"'] \arrow[r, phantom, "\top"] \arrow[d, phantom, "\dashv"] & \mathbf{EED} \arrow[l, shift right, "U"'] \arrow[d, shift right, "(-)_{\mathrm{ex/eed}}"'] \arrow[d, phantom, "\dashv"] & \mathbf{LVS} \arrow[l, hook'] \arrow[ddl, bend right, "(-)_{\mathrm{ex/lvs}}"'] \arrow[ddl, phantom, "\vdash"] & \\
 & & & \mathbf{PrWEq} \arrow[ull] \arrow[ul, hook] \arrow[dl, hook] \arrow[dll] \\
\mathbf{LEX} \arrow[r, shift right, "(-)_{\mathrm{ex/lex}}"'] \arrow[u, shift right, "\mathrm{Sub}"'] \arrow[r, phantom, "\top"] & \mathbf{EX} \arrow[l, shift right, "U"'] \arrow[u, shift right, "\mathrm{Sub}"'] \arrow[uur, "U"'] & \mathbf{WLEX} \arrow[l, "(-)_{\mathrm{ex/wlex}}"] \arrow[uu, hook, "J"] &
\end{tikzcd}$$

A diagram of the talk

$$\begin{tikzcd}
\mathbf{ED} \arrow[r, shift right=1.5ex, "(-)_{\mathrm{E}}"'] \arrow[r, phantom, "\top"] \arrow[dd, shift right=1.5ex, "(\int -)_{\mathrm{x}}"'] \arrow[dd, phantom, "\dashv"] & \mathbf{EED} \arrow[l, shift right=1.5ex, "U"'] \arrow[dd, shift right=1.5ex, "(-)_{\mathrm{ex/eed}}"'] \arrow[dd, phantom, "\dashv"] & \mathbf{LVS} \arrow[l, hook'] \arrow[ddl, bend right] \arrow[ddl, phantom, "\perp"] & \\
 & & & \mathbf{PrWEq} \arrow[uull] \arrow[uul, hook'] \arrow[ddl, hook] \arrow[ddll] \\
\mathbf{LEX} \arrow[r, shift right=1.5ex, "(-)_{\mathrm{ex/lex}}"'] \arrow[r, phantom, "\top"] \arrow[uu, shift right=1.5ex, "\mathrm{Sub}"'] & \mathbf{EX} \arrow[l, shift right=1.5ex, "U"'] \arrow[uu, shift right=1.5ex, "\mathrm{Sub}"'] \arrow[uur, "U"'] & \mathbf{WLEX} \arrow[l, "(-)_{\mathrm{ex/wlex}}"] \arrow[uu, hook, "J"] &
\end{tikzcd}$$

The doctrine of local variations on the site $(\mathbf{B}, \sigma)$

$\mathbf{B}$ is a category with finite products and weak equalisers

σ is a Grothendieck topology on $\mathbf{B}$ generated by singleton covers

The doctrine $\mathrm{Vrn}_{\mathbf{B},\sigma}\colon \mathbf{B}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[dd, "f"] & \mathrm{Vrn}_{\mathbf{B},\sigma}(X) \arrow[dd, shift right=2, "f^*"'] \arrow[dd, phantom, "\dashv"] \\
f \arrow[r, mapsto] & {} \\
X' \arrow[r, mapsto] & \mathrm{Vrn}_{\mathbf{B},\sigma}(X') \arrow[uu, bend right, shift right=2, "f\circ - = \exists_f"']
\end{tikzcd}$$

where the fibre $\mathrm{Vrn}_{\mathbf{B},\sigma}(X)$ is the poset reflection of

$$\begin{tikzcd}
V \arrow[d] \arrow[rr] & & B \arrow[dl, "\beta"] \\
A \arrow[r, "\alpha"'] & X &
\end{tikzcd}$$

Theorem

The doctrine $\mathrm{Vrn}_{\mathbf{B},\sigma}\colon \mathbf{B}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is elementary existential

It is the associated σ-sheaf of the doctrine of variations $\mathrm{Vrn}_{\mathbf{B}}\colon \mathbf{B}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$

The doctrine of local variations on the site $(\mathbf{B}, \sigma)$

$\mathbf{B}$ is a category with finite products and weak equalisers

σ is a Grothendieck topology on $\mathbf{B}$ generated by singleton covers

The doctrine $\mathrm{Vrn}_{\mathbf{B},\sigma} \colon \mathbf{B}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f", ""{name=F}] & \mathrm{Vrn}_{\mathbf{B},\sigma}(X) \arrow[d, bend right, "f^*"', ""{name=G}] \arrow[d, phantom, "\dashv"] \\
X' \arrow[r, mapsto] & \mathrm{Vrn}_{\mathbf{B},\sigma}(X') \arrow[u, bend right, "f \circ - = \exists_f"']
\arrow[from=F, to=G, mapsto]
\end{tikzcd}$$

where the fibre $\mathrm{Vrn}_{\mathbf{B},\sigma}(X)$ is the poset reflection of

$$\begin{tikzcd}
V \arrow[d] \arrow[rrd] & & \\
A \arrow[rd, "\alpha"'] & & B \arrow[ld, "\beta"] \\
& X &
\end{tikzcd}$$

Corollary

For $\mathbf{R}$ a regular category and reg the regular topology on $\mathbf{R}$

$$\mathrm{Vrn}_{\mathbf{R},\mathrm{reg}} = \mathrm{Sub}_{\mathbf{R}} \colon \mathbf{R}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$$

The 2-category $\mathbf{LVS}$

0-cells are sites $(\mathbf{B},\sigma)$ where $\mathbf{B}$ has finite products and weak equalisers and σ is generated by singleton covers

1-cells are functors $F\colon\mathbf{B}\longrightarrow\mathbf{B}'$ which take covers in σ to covers σ' and induce a homomorphism of e.e.d.s from $\mathrm{Vrn}_{\mathbf{B},\sigma}$ to $\mathrm{Vrn}_{\mathbf{B}',\sigma'}$

2-cells are natural transformations $\eta\colon F\overset{\cdot}{\longrightarrow}G$ which induce a 2-cell from $\mathrm{Vrn}_{\mathbf{B},\sigma}$ to $\mathrm{Vrn}_{\mathbf{B}',\sigma'}$

Examples

$F\colon\mathbf{C}\longrightarrow\mathbf{C}'$ for $\mathbf{C}$ and $\mathbf{C}'$ with finite products and weak equalisers is a 1-cell in $\mathbf{LVS}$ if and only if F preserves finite products and weak equalisers

$F\colon\mathbf{C}\longrightarrow\mathbf{R}$ from $\mathbf{C}$ with finite products and weak equalisers to $\mathbf{R}$ regular is a 1-cell in $\mathbf{LVS}$ if and only if F preserves finite products and takes a weak equaliser of $C\rightrightarrows D$ to a regular cover of an equaliser of $FC\rightrightarrows FD$

The 2-functors $\mathbf{EX} \xrightarrow{U} \mathbf{LVS} \xleftarrow{J} \mathbf{WLEX}$

For $\mathbf{E}$ an exact category $\qquad U(\mathbf{E}) = (\mathbf{E}, \mathrm{reg})$

For $\mathbf{C}$ a category with weak limits $\qquad J(\mathbf{C}) = (\mathrm{FinPrd}(\mathbf{C}), \pi)$

- $\mathrm{FinPrd}(\mathbf{C})$ is the completion of $\mathbf{C}$ with respect to finite products

$$(C_i)_{i\in I} \xrightarrow{\left(\ell\colon J \to I, (f_j\colon C_{\ell(j)} \to D_j)_{j\in J}\right)} (D_j)_{j\in J}$$

- π is the topology on $\mathrm{FinPrd}(\mathbf{C})$ generated by the singletons

$$\left\{ (W) \xrightarrow{\left(!\colon I \to 1, (p_i\colon W \to C_i)_{i\in I}\right)} (C_i)_{i\in I} \right\}$$

such that the diagram
$$\begin{tikzcd}
 & W \arrow[dl, "p_{i_1}"'] \arrow[dr, "p_{i_n}"] & \\
C_{i_1} & \cdots & C_{i_n}
\end{tikzcd}$$
is a weak product in $\mathbf{C}$

The completion of the solution

Lemma

The fibre $\mathrm{Vrn}_{J(\mathbf{C})}\big((C_i)_{i\in I}\big)$ is the poset reflection of $\mathbf{C}/C_{i_1},\dots,C_{i_n}$

$$\begin{tikzcd}
A \arrow[rrrr] \arrow[dr, "a_1"'] \arrow[drrr, "a_n" near start] \arrow[drr, phantom, "\cdots" near start] & & & & B \arrow[dl, "b_n"] \arrow[dlll, "b_1"' near start] \arrow[dll, phantom, "\cdots" near start] \\
& C_{i_1} & \cdots & C_{i_n} &
\end{tikzcd}$$

J. Emmenegger. On the local cartesian closure of exact completions. *JPAA*, 2020

C.Jr. Cioffo. Biased elementary doctrines and quotient completions. *JPAA*, to appear

The completion of the solution

Theorem

Let $\mathbf{C}$ be with weak finite limits
A functor $F\colon \mathbf{C} \longrightarrow \mathbf{R}$ to a regular category is left covering if and only if
its extension $\hat{F}\colon \mathrm{FinPrd}(\mathbf{C}) \longrightarrow \mathbf{R}$ is a 1-cell $(\mathrm{FinPrd}(\mathbf{C}), \pi) \longrightarrow (\mathbf{R}, \mathrm{reg})$ in $\mathbf{LVS}$
Hence

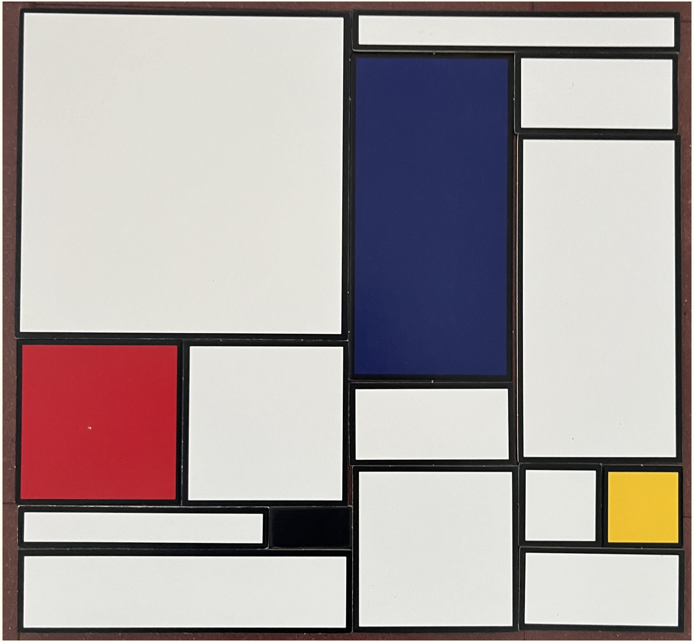
$$\mathbf{EX}(\mathbf{C}_{\mathrm{ex/wlex}}, \mathbf{E}) \equiv \mathrm{LftCov}(\mathbf{C}, \mathbf{E}) \equiv \mathbf{LVS}(J(\mathbf{C}), U(\mathbf{E}))$$

Theorem

The exact completion $(-)_{\mathrm{ex/wlex}}\colon \mathbf{WLEX} \longrightarrow \mathbf{EX}$ extends to a biadjunction

$$\begin{tikzcd}
 & \mathbf{LVS} \arrow[dl, shift right=2, "(-)_{\mathrm{ex/lvs}}"'] \arrow[dl, phantom, "\perp"] \\
\mathbf{EX} \arrow[ur, shift right=2, "U"'] & \mathbf{WLEX} \arrow[l, "(-)_{\mathrm{ex/wlex}}"] \arrow[u, hook, "J"']
\end{tikzcd}$$

The diagram at the end of the talk



The diagram at the end of the talk

$$\begin{tikzcd}
\mathbf{ED} \arrow[r, shift right, "(-)_{\mathrm{E}}"'] \arrow[r, phantom, "\top"] \arrow[dd, shift right, "(\int -)_{\mathrm{x}}"'] \arrow[dd, phantom, "\dashv"] & \mathbf{EED} \arrow[l, shift right, "U"'] \arrow[dd, shift right, "(-)_{\mathrm{ex/eed}}"'] \arrow[dd, phantom, "\dashv"] & \\
 & & \mathbf{LVS} \arrow[ul, hook] \arrow[dl, shift right, "(-)_{\mathrm{ex/lvs}}"'] \arrow[dl, phantom, "\perp"] \\
\mathbf{LEX} \arrow[uu, shift right, "\mathrm{Sub}"'] \arrow[r, shift right, "(-)_{\mathrm{ex/lex}}"'] \arrow[r, phantom, "\top"] & \mathbf{EX} \arrow[l, shift right, "U"'] \arrow[uu, shift right, "\mathrm{Sub}"'] \arrow[ur, shift right, "U"'] & \mathbf{WLEX} \arrow[u, hook, "J"'] \arrow[l, "(-)_{\mathrm{ex/wlex}}"]
\end{tikzcd}$$

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