Enriched Coalgebras and Coinductive Predicates

Henning Basold*

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Dedicated to William Lawvere

Coalgebra has emerged from the desire to find an abstraction of the behaviour of computational models [Rut00]. The main idea is that systems can be modelled as morphisms $c\colon X \to FX$ in some category $\mathcal{C}$, where $F\colon \mathcal{C} \to \mathcal{C}$ determines the type of observation that can be made on the systems. A particularly important coalgebra, if it exists, is a final object in the category $\mathrm{CoAlg}(F)$ of coalgebras and their morphisms, in which the behaviour of coalgebras can be compared. Here, the main device of comparison is equality, which often coincides with bisimilarity [Sta11]. However, more general predicates are often of interest. One instance of this is coalgebraic modal logic [Cîr+11; Mos99], in which modalities are added to a basic logic that enable reasoning about the possible observations in a coalgebra. The picture that emerges from this is that coinductive predicates for coalgebras of type F are modelled as coalgebras for a fibrational lifting G of F [KR21]. This situation can be represented by a fibration of coalgebras as follows. Suppose that $(F, G)\colon p \to p$ is a fibration morphism. Then there is a fibration of coalgebras with a forgetful fibration morphism (U_F, U_G) as in the following diagram.

$$\begin{array}{ccccc}
\mathbb{E} & \xrightarrow{G} & \mathbb{E} & \xleftarrow{U_G} & \mathrm{CoAlg}(G) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
\mathbb{B} & \xrightarrow{F} & \mathbb{B} & \xleftarrow{U_F} & \mathrm{CoAlg}(F)
\end{array}$$

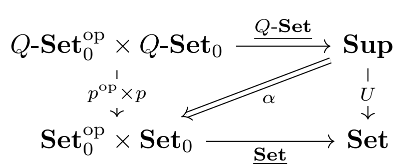
The objects in $\mathrm{CoAlg}(G)$ are pairs triples (c, P, r) where $c\colon X \to FX$, P is an object in the fibre $\mathbb{E}_X$ above X and $r\colon P \to (c^* \circ G)P$ in $\mathbb{E}_X$. Morphisms $(c, P, r) \to (d, Q, s)$ are pairs (f, a) of a coalgebra morphism $f\colon c \to d$ in $\mathrm{CoAlg}(F)$ and a morphism $a\colon P \to Q$ with $pa = f$, such the following diagram commutes for $\mathrm{tr}^c_d a$ a transport of a from c to d defined in terms of Cartesian liftings.

$$\begin{array}{ccc}
P & \xrightarrow{a} & Q \\
{\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\
c^*GP & \xrightarrow{\mathrm{tr}^c_d a} & d^*GQ
\end{array}$$

*Leiden University, LIACS; h.basold@liacs.leidenuniv.nl

Frequently, coalgebras and coalgebraic predicates come with extra structure in form of enrichment that arises in several forms: the base category $\mathcal{C}$ is itself already enriched [Wil13], the functor F can create enrichment in the Kleisli category, the total category of the fibration is enriched, or even the whole fibration is enriched. We will focus in this presentation on the first three options, as they seem to be the most frequent. A use case for an enriched base category are coalgebras on vector spaces that are **Ab**-enriched or we may study coalgebra with homotopical information by using a simplicial category as base category. This can even be furthered to a study of homotopical categories [Dwy+05; Rie14] or model categories of coalgebras as a generalisation of work in concurrency [Dub17; Eri03; Faj+16; Gau21] and on hybrid systems [AS05]. The second option, the creation of enriched Kleisli categories, is common in semantics, where one uses a monad to model side effects together with a **CPO**-enrichment of the Kleisli category to model recursion. Finally, the third option, where only the total category is enriched, is quite similar to the previous option. The functor F may carry some more structured information than the base category. This occurs, for example, when probabilistic systems are modelled as coalgebras for a functor on **Set** and we wish to reason about them with a quantitative logic that can be modelled in a suitable fibration of quantitative sets. Another possible application would be reasoning about systems with geometrical in terms of modal logic [VdGB22; vDit+20; vDK22].

In the presentation, I would like to show how enrichment of coalgebras and Kleisli categories work and what their use cases are. We will then study situations where only the total category of a fibration has interesting enrichment. In particular, we will show that there is a category of Q-**Set** of sets with equality valued in a quantale Q that is enriched over the category **Sup** of complete lattices and sup-preserving maps, while forming a fibration $p\colon Q\text{-}\mathbf{Set}_0 \to \mathbf{Set}$. The construction of Q-**Set** follows that of H-**Set** over a Heyting algebra H [FS79; Hig84] and the **Sup**-enrichment is similar to that of Q-**Rel** [HST14]. Unfortunately, this situation does not fit any of the recent definitions of enriched fibrations[BW19; Shu13; Vas18] and we have to find a more general approach that is still well-behaved. For example, the situation that we find the in case of Q-**Set** is that the forgetful functor $U\colon \mathbf{Sup} \to \mathbf{Set}$ is (lax) monoidal and that there is a natural transformation α as in the following diagram, where $\underline{Q\text{-}\mathbf{Set}}$ and $\underline{\mathbf{Set}}$ are the enriched hom-functors, where we see **Set** as self-enriched.



In the course of the presentation, I will present some preliminary results and open questions that come out of this ongoing work.

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