The fibration of algebras

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Let $\mathcal{A}$ be a category with finite products: each object A in $\mathcal{A}$ induces a comonad $T_A = \_ \times A : \mathcal{A} \to \mathcal{A}$. The coKleisli category of this comonad consists of the so-called *simple slice* $\mathcal{A}/\!\!/A$ over A; its objects are the same as $\mathcal{A}$, and a coKleisli morphism is a map $f : X \times A \to Y$ in $\mathcal{A}$ – each such morphism can be regarded as an abstract family of maps $f_a : X \to Y$, parametrized by A, and in fact this is precisely what happens if $\mathcal{A} = \mathsf{Set}$, as $f : X \times A \to Y$ transposes to a function $A \to Y^X$. Clearly there is a covariant functor $\mathcal{A} \to [\mathcal{A}, \mathcal{A}]$ sending $A \mapsto T_A$, while the association $A \mapsto \mathcal{A}/\!\!/A = \mathsf{coKl}(T_A)$ is contravariant. The simple slice construction is fundamental in categorical logic because of its role in the foundation of semantics of simple type theory as introduced by Church in [2].

Now, consider again a Cartesian category $(\mathcal{A}, \times)$; any internal monoid M defines a monad $M \times \_$ on $\mathcal{A}$ whose Eilenberg-Moore category is precisely the category of objects with an action of M. If we let M vary over the entire category $\mathsf{Mon}(\mathcal{A})$ of internal monoids in $\mathcal{A}$ we recover an old friend of algebraic geometers and algebraic topologists [1, 4, 8].

Consider, now, the following well-known theorem in the theory of representations of Hopf algebras:

> [Cartier-Gabriel-Konstant] A cocommutative Hopf algebra over $\mathbb{K}$ is the semidirect product of a group acting on a Lie algebra over $\mathbb{K}$.

Our work aims to provide a common framework in which all these seemingly disconnected examples, and many others, fit naturally and can be studied in a uniform way. The key idea is to consider a general theory of parametric endofunctors $A \mapsto T_A$ arising from functors of type

$$T : \mathcal{A} \to [\mathcal{X}, \mathcal{X}]$$

or, equivalently, of type $\mathcal{A} \times \mathcal{X} \to \mathcal{X}$, with a particular focus on the way in which we associate to T_A a category of algebras.

'Algebras' here has to be intended as broadly as the structure on T_A allows: if T_A is a co/monad, it will be natural to attach a co/Eilenberg-Moore category to it; if T_A is a mere pointed endofunctor, it will be natural to consider its pointed endofuctor algebras. The key observation now is that the association $T_\bullet : A \mapsto \mathsf{Alg}(T_A)$ is a pseudofunctor $\mathcal{A} \to \mathsf{Cat}$, and as such it defines a fibration (when $T_\bullet$ is contravariant) or an opfibration (when covariant) over $\mathcal{A}$. Here the fiber over a given A is exactly the category of T_A-algebras.

Therefore this is the subject of our study: *fibrations and opfibrations associated to parametric endofunctors of sorts*. We aim at building a general theory of such constructions, outlining the common properties shared by a seemingly scattered variety of examples, and what structure theorems one can have available once assuming just a bit more about $\mathcal{A}$, on $\mathcal{X}$, or on the subcategory of $[\mathcal{X}, \mathcal{X}]$ through which T factors (for example: $\mathcal{X}$ might be κ-accessible, and each T_A an endofunctor commuting with κ-filtered colimits).

Such a theory has an intrinsic interest: we provide additional evidence that parametricity as intended by French category theory between the 50s and the 80s (a family of categories continuously varying over a category of indices, a point of view championed, among others, by J. Bénabou), and parametricity studied by computer scientists (the nontrivial dependence of a functor from a set of 'states' whence different outcomes of a computation arise), are two manifestations of the same general principle. We recollect examples from a surprisingly large variety of fields of application: type theory (various classes of fibrations in categorical logic arise as fibrations of algebras, and polynomial functors all give rise to a fibration of their endofunctor algebras), topos theory (the Kelly-Lawvere lattice of essential localizations), representation theory, algebraic topology, categorical algebra (Lawvere hyperdoctrines), computer science (with particular attention to *dinatural* parametricity), and more.

We hope that our work will be of interest to both communities, and especially to the growing boundary region between the two.

As category theorists, we believe the most fruitful way to present this story is providing two different keys to the reader.

The *analytic* perspective focuses on the specifics of the fibrations of algebras, on concrete examples, and on structural theorems of sort. This way of presenting the theory largely relies on fibered category theory.

The *synthetic* perspective, on the other hand, adopts a more formal category-theoretic style, and is based on the fact that a parametric endofunctor $T : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ can evidently be thought of as an algebra in its own right, an algebra for the endo-2-functor $\mathcal{A} \times \_ : \mathsf{Cat} \to \mathsf{Cat}$ or, equivalently, as an Eilenberg-Moore algebra for the monad $\mathcal{A}^* \times \_$ induced by the free monoidal category on $\mathcal{A}$. At the same time this

- provides a clear connection with the theory of *graded monads*, [9, 6, 3, 5, 7]: the theory of fibrations of algebras can be thought of as the theory of graded monads over a 'free grading monoid';
- provides an enticing interpretation of our theory as a form of *categorified semidirect product*: the total category of the fibration of algebras arising from $T : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ exhibits many similarities with a product of type $\mathcal{A} \ltimes_T \mathcal{X}$; we explore this point of view with particular focus on the possibility of building a category $\mathsf{Ext}(\mathcal{A}, \mathcal{X})$ of 'extensions of $\mathcal{X}$ by $\mathcal{A}$', in which fibrations of algebras form a well-behaved subcategory.

References

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