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Strictifying Path Categories

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A dependent type theory is said to have *propositional identity types* if it is endowed with a type constructor satisfying the usual rules of intensional identity types, but the computation rule, which is only required to hold in a weakened form. In detail, whenever we are given judgements $x, y : A;\ p : x = y \vdash C(x, y, p) : \textsc{Type}$ and $x : A \vdash q(x) : C(x, x, r(x))$, in place of asking that the judgemental equality $x : A \vdash J(x, x, r(x), q) \equiv q(x)$ holds, we only ask that it holds *propositionally*, i.e. that the identity type:

$$x : A \vdash J(x, x, r(x), q) = q(x)$$

is inhabited (here J denotes the identity type eliminator); see [1,3] for more details.

In [3] van den Berg introduces the notion of *path category* as a possibility to phrase the semantics of a dependent type theory with propositional identity types, showing how the syntax of such a *propositional* type theory can be naturally made into such a structure. However, similarly to the case of locally cartesian closed categories and general comprehension categories, it is not completely clear in what sense a path category is a model of propositional type theory, since the interpretation of a re-indexing of a type corresponds to the choice of a pullback square inside the given category, a choice that in general is not split.

In this talk we compare the notion of path category to the one of comprehension category and show how path categories can be naturally characterised as Grothendieck fibrations carrying a structure of comprehension category with propositional identity types and strong dependent sum types. Secondly, we take advantage of this characterisation and of the results on the left adjoint splitting contained in [2,4] to deduce that every path category is equivalent -in the sense of comprehension categories- to an actual (i.e. split) model of propositional type theory: a model that interprets type-theoretic substitution strictly and whose type constructors (propositional identity types and strong dependent sum types) are strictly stable under re-indexing.

References

[1] Coquand, Danielsson. Isomorphism is equality. 2013.

[2] Lumsdaine, Warren. The local universes model: an overlooked coherence construction for dependent type theories. 2015

[3] van den Berg. Path categories and propositional identity types. 2018.

[4] Bocquet. Strictification of weakly stable type-theoretic structures using generic contexts. 2021.

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