The $(\infty, 2)$-categorical externalization functor*

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The theory of presheaves – that is, the theory of indexed (or fibered) sets over a category $\mathcal{B}$ – is an extraordinarily essential part of the practice of virtually any branch of category theory. The arguably most well-known notion in this context is the Yoneda embedding, and righteously so in virtue of its plethora of crucial aspects and properties. Amongst others, whenever we start with a left exact category $\mathcal{B}$, it allows us to translate freely between (finite) limit structures associated to the logic of $\mathcal{B}$ and (finite) limit structures associated to the logic of sets indexed over $\mathcal{B}$. It canonically generates the category $\hat{\mathcal{B}}$ of presheaves, and in terms of the Yoneda Lemma determines the representable presheaves by a universal property.

Accordingly, the theory of indexed (or fibered) categories over another category $\mathcal{B}$ is, although less pervasive in the textbook literature, as essential to many branches of mathematics ranging from geometry to logic. Just as its discrete counterpart, it comes with a version of a Yoneda embedding (which in fact is directly induced by it) called the externalization functor. If we start again with a left exact category $\mathcal{B}$, the externalization functor

$$\mathrm{Ext}\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Fun}(\mathcal{B}^{op}, \mathrm{Cat})$$

maps internal categories in $\mathcal{B}$ to indexed categories over $\mathcal{B}$ by applying the Yoneda embedding pointwise. It is studied in some depth in Jacobs' textbook [2, Section 7.3], but appears to be otherwise fairly underrepresented (case in point it barely has an nlab page!).

To do it due justice, in this talk we study the externalization functor in the context of ∞-category theory. The aim is to prove some of the central results of Jacobs in this context, to show some results that seem to not have been yet considered, and to show some others that are intrinsically ∞-categorical. The take-away is that the externalization functor exhibits surprisingly many parallels to the Yoneda embedding, and hence allows for the study of internalization (and externalization) of much more involved categorical structures than the discrete Yoneda embedding does. Indeed, it faithfully extends the Yoneda embedding in as much as the square of solid arrows

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "y"] \arrow[d, hook, bend left] \arrow[dr, phantom, "\lrcorner"] & \hat{\mathcal{B}} \arrow[d, hook, bend left] \\
\mathrm{Cat}_\infty(\mathcal{B}) \arrow[r, "\mathrm{Ext}"'] \arrow[u, dashed, "(\cdot)^{\simeq}"] & \mathrm{Fun}(\mathcal{B}^{op}, \mathrm{Cat}_\infty) \arrow[u, dashed, "(\cdot)^{\simeq}"]
\end{tikzcd}$$

is a pullback. We show that the functor Ext has its own Yoneda Lemma and is in particular fully faithful. In fact, it is naturally a functor of $(\infty, 2)$-categories, and will be exhibited as a limit preserving embedding of such in the case that the base $\mathcal{B}$ is presentable (rather for simplicity than necessity however).

Furthermore, as worked out in the preprint [4], the accordingly representable indexed ∞-categories over $\mathcal{B}$ in the image of the Ext-functor are exactly the "small" ones in the sense of Bénabou's comprehension schemes ([1], [3, B1.3]). The same statement is not true in the ordinary categorical case but has

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to be remedied with some caveats because of the typical mismatch between equality and equivalence. It follows that such representable indexed ∞-categories satisfy all finite comprehension schemes which implies the internalizability of all sorts of diagrammatic structures. Furthermore, we can show that the universal cartesian fibration is representable, which among others implies a close relationship between representability and right adjointness.

As time permits, we will give an example of the usefulness of the Yoneda Lemma for indexed ∞-categories in the context of higher topos theory, where we can use the lemma to construct a sequence of object classifiers in an ∞-topos $\mathcal{B}$ which represents the target fibration of $\mathcal{B}$ not only piecewise but functorially (this allows for example for a "polymorphic" definition of higher Lawvere-Tierney operators).

References

[1] Bénabou, J. Fibered Categories and the Foundations of naive Category Theory. *The Journal of Symbolic Logic*, 1985, Volume 50, Number 1, pages 10-37.

[2] Jacobs, B. Categorical Logic and Type Theory. *Elsevier Science B.V.*, 1999, Volume 141, Studies in Logic and the Foundations of Mathematics.

[3] Johnstone, P.T. Sketches of an Elephant: A Topos Theory Compendium Volume 2. *Clarendon Press*, 2002. Number of pages 730.

[4] Stenzel, R. $(\infty, 1)$-Categorical comprehension schemes. Preprint: arxiv.org/abs/2010.09663 (2020).