Eliminating imaginaries and adding structure: how does the geometric completion fit in?

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Overview. Broadly speaking, there are two kinds of completions of doctrines considered in the literature. There are those completions which add structure to the fibres, such as Trotta's existential completion [9] or Coumans' canonical completion [5]. And then there are those which add structure to the indexing category, exemplified in the work [6].

In this talk we will present the *geometric completion* of a doctrine - a completion that adds geometric structure to the fibres of a doctrine, and study its interaction with the exact completions of doctrines.

The geometric completion. *Geometric doctrines* are those doctrines whose internal language is geometric, i.e. it interprets the symbols $\{ \wedge, \exists, =, \bigvee \}$. They possess a strong link with (pointfree) topology since they are precisely the *internal locales* of Grothendieck toposes. The geometric completion, an application of the fibred ideal completion established in [1] and described in [10], constitutes the universal completion of a doctrine to a geometric one. Importantly, the geometric completion takes a Grothendieck topology as a second argument, and because of this it can be made into an idempotent completion.

Eliminating imaginaries. Many of the completions considered in categorical logic, e.g. the 'tripos' completion, the pretopos completion etc., share a flavour of Shelah's *elimination of imaginaries* in that they seek to add objects that interpret partial equivalence relations. As observed by Makkai, the pretopos completion of a logical category is the corresponding categorification Shelah's construction for finitary first-order logic.

A unified account of the exact completions of categories is formalised in [8] and [7]. These completions exist on a spectrum from the exact completion, where we add an object for each partial equivalence relation, to the infinitary pretopos completion, where we add an object for each family of partial equivalence relations.

Having described the geometric completion, we will give a unified account of the exact completions of a geometric doctrine (over a cartesian category) similar to the approach for categories given in [8] and [7]. Since geometric doctrines have a sufficiently rich internal language, we can explicitly construct everything from the 'tripos' completion to its infinitary familial version. We will observe that the infinitary 'tripos' completion of a geometric doctrine coincides with the topos of its *internal sheaves* (when the doctrine is viewed as an internal locale).

Removing structure. There is significant interest in describing exact completions for weaker and weaker initial structures. For example, after describing the exact completion of a category with finite limits in [2], Carboni joined forces with Vitale in [3] to describe the exact completion of a category with only *weak* finite limits. This interest has been paralleled by recent work for doctrines, notably in the work of Cioffo [4].

The geometric completion exists at the extreme end of this spectrum in that it can be described for entirely *unstructured* doctrines (that is, any preorder-valued functor). Using a relative topos-theoretic approach, we will construct a general framework for describing the 'tripos'-like completions of the geometric completion of a doctrine that includes both:

(i) completions of the indexing category (without data from the internal language of the doctrine),

(ii) exact completions that add (familial) partial equivalence relations to the indexing category.

This presentation represents work in progress.

References

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