First steps in algebraic type theory

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A type theoretic universe $u : \mathcal{U}^\star \to \mathcal{U}$ bears an algebraic structure that represents the type-forming operations in the fibration of (dependent) types that it classifies. For example, the polynomial composite

$$u.u : \mathcal{U}_2^\star \to \mathcal{U}_2$$

classifies types in an extended context (A, B), and the Σ type former is then represented by an operation

$$\Sigma : \mathcal{U}_2 \to \mathcal{U}$$

which together with the term-former

$$\mathsf{pair} : \mathcal{U}_2^\star \to \mathcal{U}^\star$$

forms a map of polynomials (a pullback square), $u.u \to u$.
Similar operations on $u : \mathcal{U}^\star \to \mathcal{U}$ represent the other type-formers of unit type, identity type, and dependent product. This structure can be abstracted to form the concept of a "Martin-Löf algebra". The theory of such ML-algebras can be seen as a proof-relevant version of the theory of Zermelo-Fraenkel algebras from the algebraic set theory of Joyal and Moerdijk. For instance, a free ML-algebra is then a model of type theory with special properties.