

# Algebraic Type Theory

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Workshop on Doctrines  
and Fibrations

Padova  
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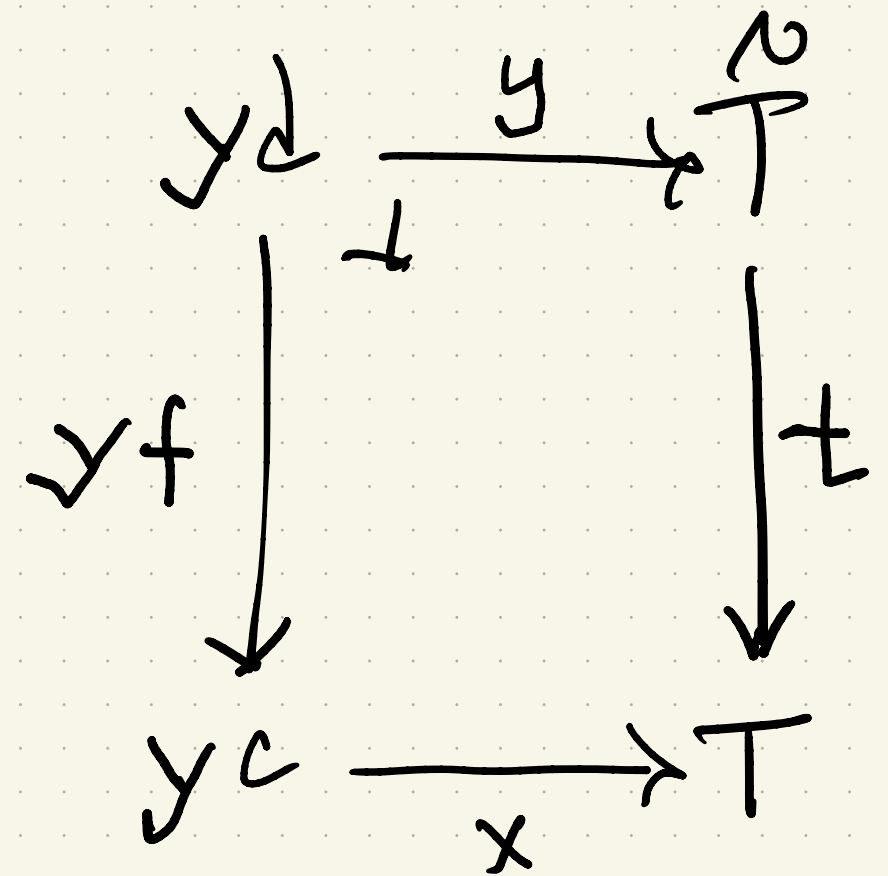
# 1. Natural Models

Def. A natural model consists of

- a category  $\mathcal{C}$
- presheaves  $T, \tilde{T}$
- a natural transformation

$$t: \tilde{T} \rightarrow T$$

- which is representable



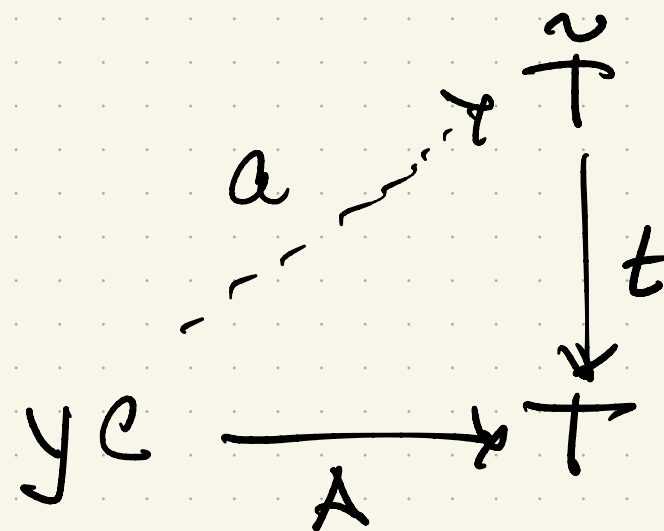
$$\forall c \in \mathcal{C} \quad \forall x \in T_c$$

$$\exists f: d \rightarrow c \quad \exists y \in \tilde{T}_d$$

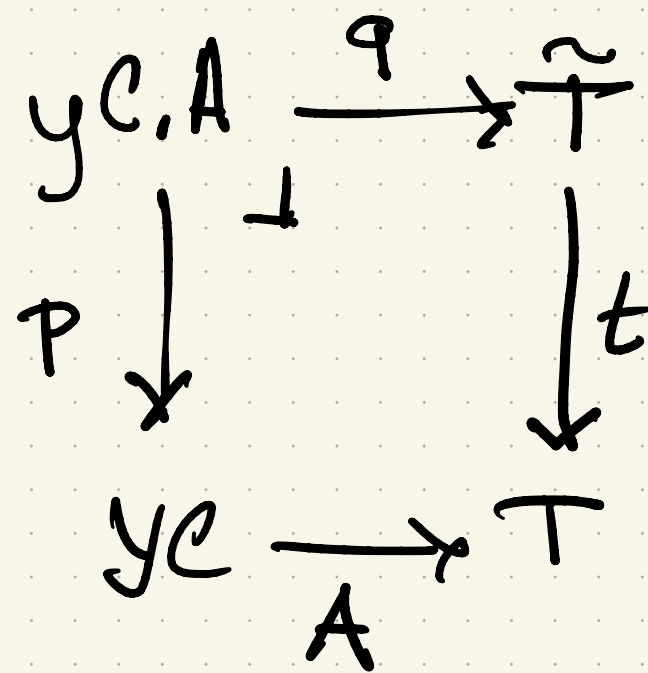
This is equivalent to CwF:

- $\mathcal{C}$  cat of contexts
- $\mathcal{T}$  presheaf of types
- $\hat{\mathcal{T}}$  presheaf of terms

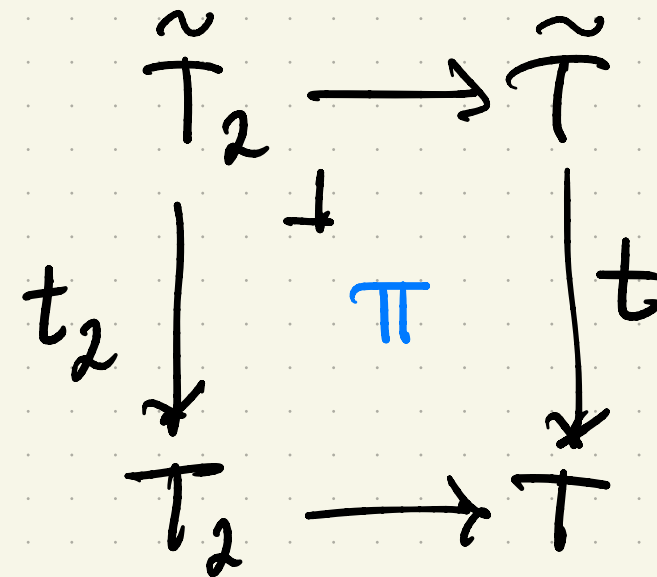
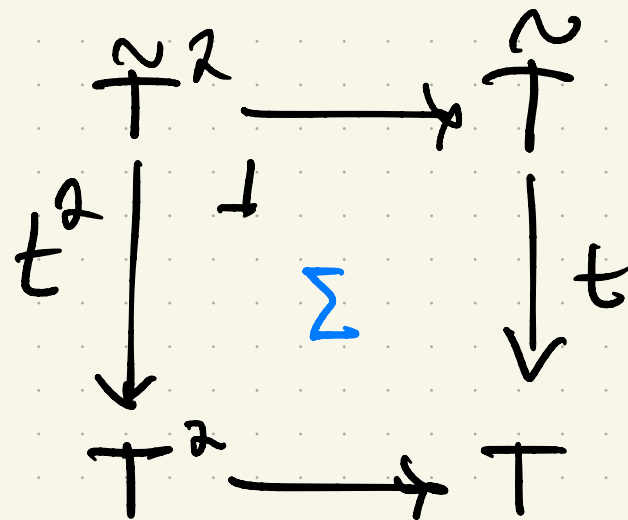
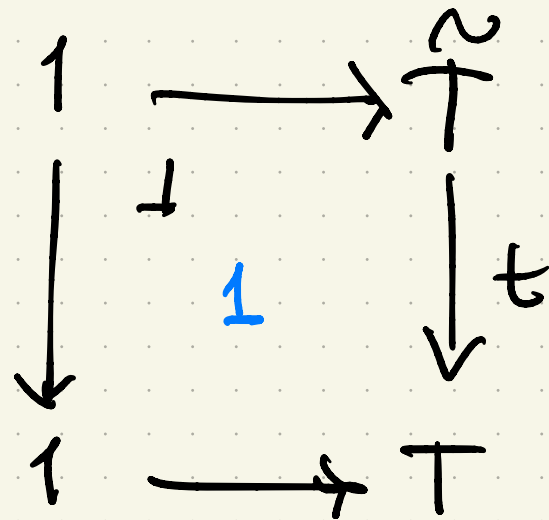
- representability is context extension.



$C \vdash a : A$



- Type formers  $\perp, \Sigma, \Pi$  are modelled by



- We will abstract this structure to form that of a Martin-Löf algebra.



## 2. Polynomial Functors

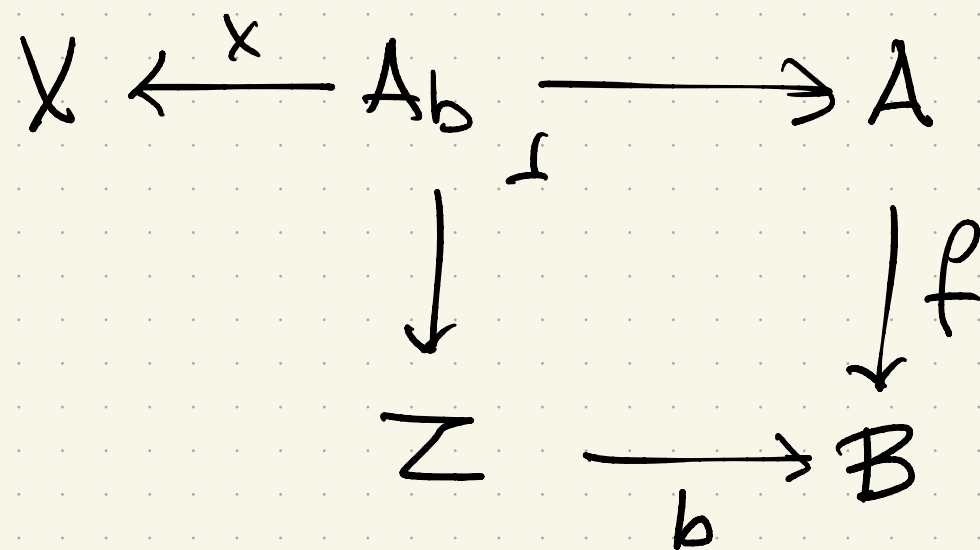
Every  $f: A \rightarrow B$  in an LCCC  $\mathcal{E}$  determines a polynomial endofunctor  $Pf: \mathcal{E} \rightarrow \mathcal{E}$ .

$$\begin{array}{ccc}
 \mathcal{E} & \xrightarrow{Pf} & \mathcal{E} \\
 A^* \searrow & & \nearrow B! \\
 \mathcal{E}/A & \xrightarrow{f_*} & \mathcal{E}/B
 \end{array}$$

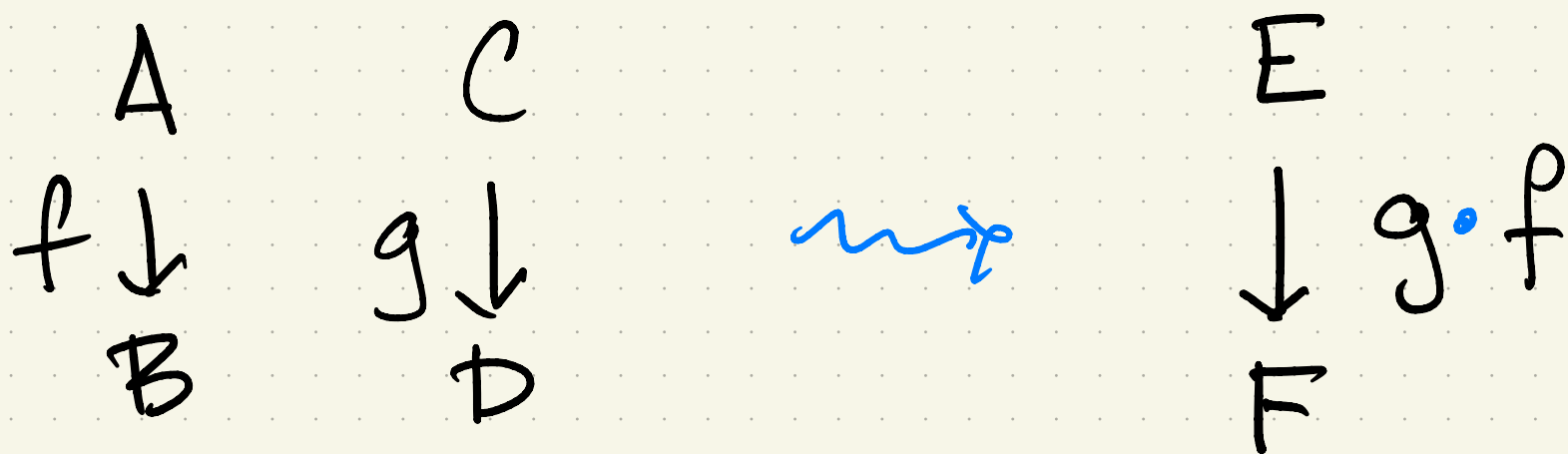
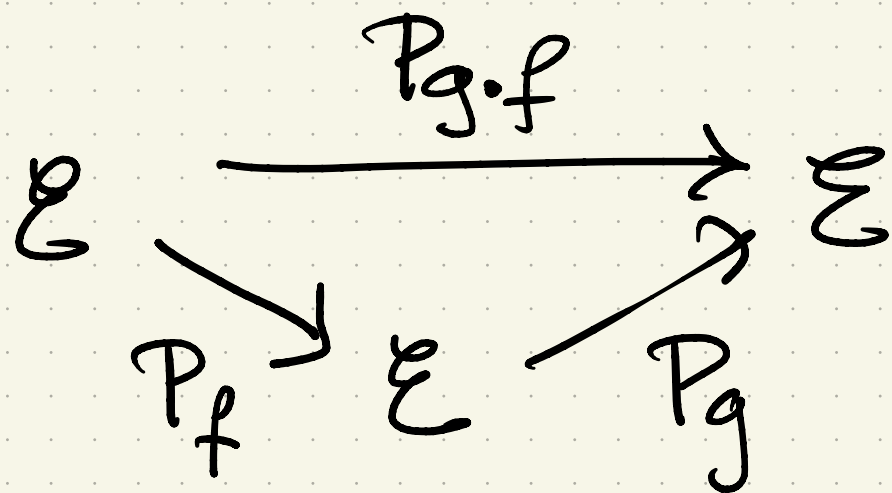
$$\begin{array}{ccc}
 X \hookrightarrow X \times A & & Pf X \\
 \downarrow & & \swarrow \\
 A & \xrightarrow{f} & B
 \end{array}$$

- In the DTT of  $\mathcal{E}$   $P_f X = \sum_{b \in B} X^{A_b}$ .

- The UMP of  $P_f X$  is  $(b, x): Z \longrightarrow P_f X$



- The composite of polynomial functors is polynomial:



- As is  $1_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{E}$ , so there is a monoid:

$$(\text{Poly } \mathcal{E}, \cdot, 1_{\mathcal{E}})$$

### 3. M-L Algebras

Def A M-L algebra in a LCC  $\mathcal{E}$  is a map

$$t: \tilde{T} \rightarrow T$$

with structure

$$\begin{array}{ccc} 1 & \xrightarrow{\sim} & \tilde{T} \\ \downarrow & u & \downarrow t \\ 1 & \xrightarrow{\quad} & T \end{array}$$

$$\begin{array}{ccc} \tilde{T}^2 & \xrightarrow{\quad} & \tilde{T} \\ t^2 \downarrow & m & \downarrow t \\ T^2 & \xrightarrow{\quad} & T \end{array}$$

$$\begin{array}{ccc} \tilde{T}_2 & \xrightarrow{\quad} & \tilde{T} \\ t_2 \downarrow & c & \downarrow t \\ T_2 & \xrightarrow{\quad} & T \end{array}$$

where  $P_{t^2} = P_{t \cdot t} = P_t \circ P_t$  and  $t_2 = P_t(t)$ .

- The **unit** determines a natural transformation

$$u: 1_{\mathcal{E}} \rightarrow P_t$$

- The **multiplication** determines another one

$$m: P_t \circ P_t \rightarrow P_t$$

- The **closure** determines an algebra structure

$$c: P_t(t) \rightarrow t$$

## Dominance

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## 4. Examples

(i) A CwF  $(\mathcal{C}, t: \tilde{T} \rightarrow T)$  has  $1, \Sigma, \Pi$  as a CwF iff it is a ML-algebra in  $\hat{\mathcal{C}}$ .

(ii) Let  $t: \tilde{T} \rightarrow T$  be a ML-algebra in a LCCC  $\mathcal{E}$ .

Define a CwF  $\hat{t}: T_m \rightarrow T_y$  by mapping in,

$$\begin{array}{ccc} T_m & & \mathcal{E}(-, \tilde{T}) \\ \hat{t} \downarrow & ::= & \downarrow \mathcal{E}(-, t) \\ T_y & & \mathcal{E}(-, T) \end{array}$$

Then  $\hat{t}$  has  $1, \Sigma, \Pi$  as a CwF.

Pf: Yoneda preserves ML-algebras, then use (i).

### iii) Clans

Let  $t: \tilde{T} \rightarrow T$  be a natural model in  $\hat{\mathcal{C}}$  and define display maps  $\mathcal{D}_t$  in  $\mathcal{C}$  by:

$$\mathcal{D}_t \ni f \downarrow c \quad \Leftrightarrow \quad \begin{array}{ccc} yd & \xrightarrow{\quad} & \tilde{T} \\ yf \downarrow & \lrcorner & \downarrow t \\ yc & \xrightarrow{\quad} & T \end{array}$$

Then  $\mathcal{D}_t$  is closed under pullbacks and under

- isos & composition if  $t$  is a **dominance**,
- pushforwards if  $t$  is **closed**.

So  $(\mathcal{C}, \mathcal{D}_t)$  is a  **$\pi$ -clan** if  $t$  is a **ML-algebra**.



Conversely given a  $\Pi$ -clan  $(\mathcal{C}, \mathcal{D})$  there's a natural model  $d: \tilde{\mathcal{D}} \rightarrow \mathcal{D}$  in  $\hat{\mathcal{C}}$  that's a ML-algebra, and  $\mathcal{D} = \mathcal{D}_d$ .

Pf Let

$$\begin{array}{ccc}
 \tilde{\mathcal{D}} & & \coprod_{y \in \mathcal{D}} y \\
 d \downarrow & ::= & \coprod_{f \in \mathcal{D}} y f \downarrow \\
 \mathcal{D} & & \coprod_{y \in \mathcal{D}} y \text{cod } f
 \end{array}$$

In fact, there's an adjunction\*

$$\text{Clans} \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} \text{Nat Mod}$$

Where

$$L(\mathbb{C}, \mathcal{D}) = \perp\!\!\!\perp y \mathcal{D}$$

$$R(\mathbb{C}, t) = (\mathbb{C}, \mathcal{D}_t) \quad .$$

(iv) Finite sets In  $\mathcal{E} = \text{Set}$  let

$$\text{nat} \begin{array}{c} \mathbb{N} \\ \downarrow \\ \mathbb{N} \end{array} = \sum_n [n] \begin{array}{c} \downarrow \\ \mathbb{N} \end{array} = \begin{array}{c} \{m < n\} \subseteq \mathbb{N} \times \mathbb{N} \\ \downarrow \\ \mathbb{N} \end{array} \begin{array}{c} \swarrow \\ \mathbb{N} \end{array} P_1$$

The polynomial functor  $P_{\text{nat}} : \text{Set} \rightarrow \text{Set}$  is then

$$P_{\text{nat}}(X) = \sum_n X^n = 1 + X + X^2 + \dots$$

• unit

$$u_X: X \longrightarrow 1 + X + \dots \quad +\text{-inclusion}$$

• multiplication

$$m: P_{\text{nat}}^2 \longrightarrow P_{\text{nat}}$$

$$\begin{array}{ccc} P_2 & \xrightarrow{\tilde{m}} & \mathbb{N} \\ \downarrow \text{nat}^2 & \lrcorner & \downarrow \\ \sum_n \mathbb{N}^n = P_{\text{nat}} \mathbb{N} & \xrightarrow{m} & \mathbb{N} \end{array}$$

$$m_n: \mathbb{N}^n \longrightarrow \mathbb{N}$$

$$(k_1, \dots, k_n) \longmapsto k_1 + \dots + k_n$$

• unit

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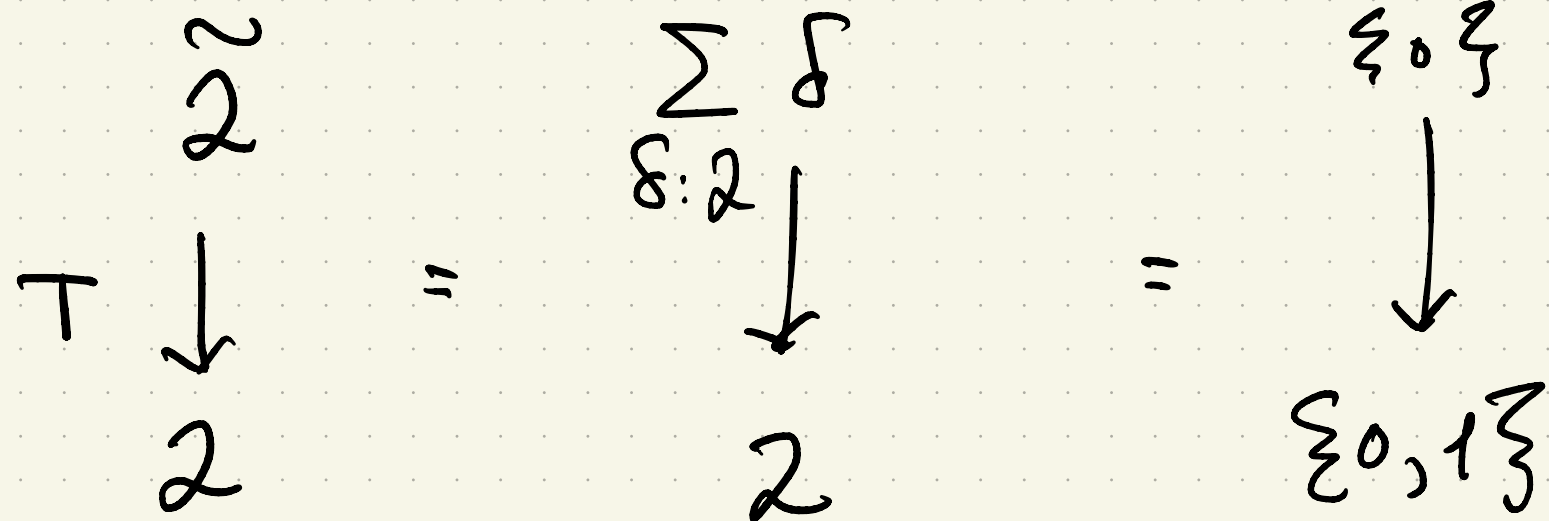
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$$m_n: \mathbb{N}^n \longrightarrow \mathbb{N}$$

$$(k_1, \dots, k_n) \longmapsto k_1 + \dots + k_n$$

• closure: exercise!

# (v) Bool



•  $P_T(X) = \sum_{\delta:2} X^\delta = 1 + X$

• unit

$$X \longrightarrow 1 + X$$

+ - inclusion

• mult

$$1 + (1 + X) \longrightarrow 1 + X$$

▽

• closure

$$\begin{array}{ccc} 1+1 & \longrightarrow & 1 \\ \downarrow & + & \downarrow \\ 1+2 & \longrightarrow & 2 \end{array}$$

SoC

## (vi) Groth Universe

Take any cardinal  $\alpha \neq \aleph_0$  do "the same thing":

$\tilde{S}_\alpha$  "  $\sum_{a \in S_\alpha} a$  "

$\downarrow$

$S_\alpha$  "sets of size  $\leq \alpha$ "

ML-algebra

if  $\alpha$  is inaccessible,

## (vii) Syntactic CwF of DTT w/ $\lambda, \Sigma, \Pi$

In Ctx:

$\text{Terms}(-) = \{ - \vdash a : A \}$

$\downarrow$

$\text{Types}(-) = \{ - \vdash A \text{ type} \}$

the initial  
ML-algebra!

## (viii) Hofmann-Streicher Universe

Given a cardinal  $\alpha \in \text{Set}$ , for any  $\mathcal{C}$ ,  
we have the  $\alpha$ -H-S universe in  $\hat{\mathcal{C}}$ :

CwF

$$U : \mathcal{C}^{\text{op}} \longrightarrow \text{Set}_\alpha$$

$$E : \int_{\mathcal{C}} U^{\text{op}} \longrightarrow \text{Set}_\alpha$$

$\cong$

$$\tilde{U}_\alpha$$

$$\downarrow$$

$$U_\alpha$$

Nat  
Mod

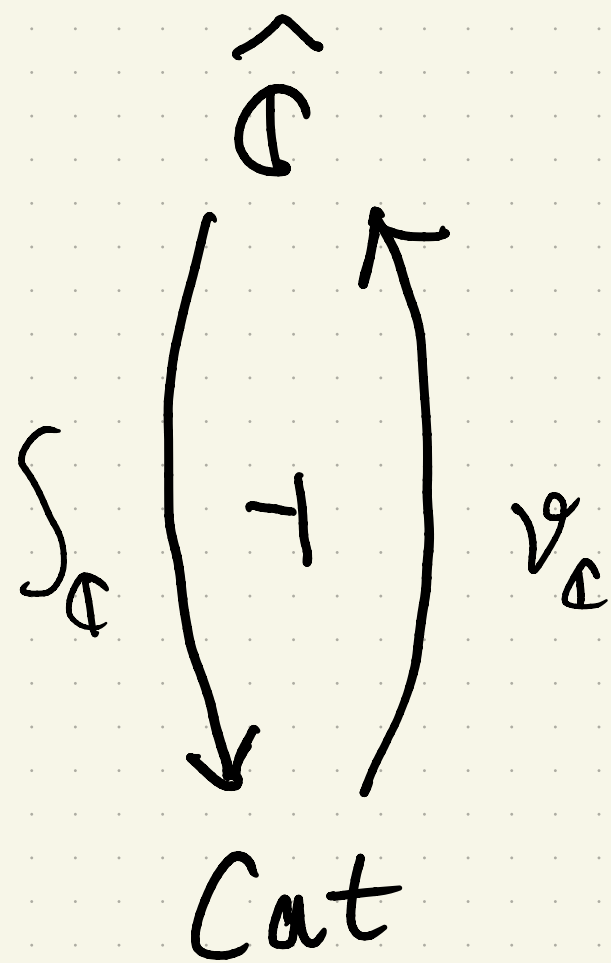
Prop.  $\tilde{U}_\alpha \rightarrow U_\alpha$  is a ML-algebra (for suitable  $\alpha$ ).

This follows from 3 facts...

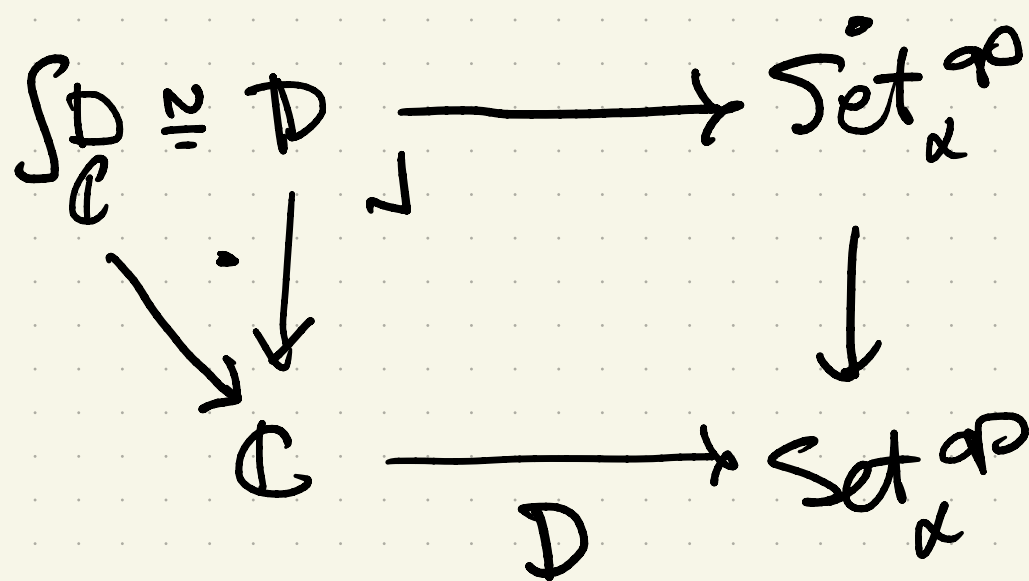


Fact 1

$\tilde{U}_\alpha \rightarrow U_\alpha$  is the nerve of the universal  $\alpha$ -small discrete fibration  $\mathring{Set}_\alpha \rightarrow Set_\alpha$  in  $Cat$ :



$$\begin{array}{ccc} \tilde{U}_\alpha & & \mathring{Set}_\alpha \\ \downarrow & = & \downarrow \\ U_\alpha & & Set_\alpha \end{array}$$



Fact 2 The nerve  $\mathcal{N}_{\mathcal{C}}: \text{Cat} \longrightarrow \hat{\mathcal{C}}$  preserves  
ML-algebras.

Fact 3  $\text{Set}_{\alpha}^{\text{op}} \longrightarrow \text{Set}_{\alpha}^{\text{op}}$  is a ML-algebra in  $\text{Cat}^*$   
(for suitable  $\alpha$ ).

So indeed:

Prop.  $\tilde{\mathcal{U}}_{\alpha} \longrightarrow \mathcal{U}_{\alpha}$  is a ML-algebra in  $\hat{\mathcal{C}}$   
(for suitable  $\alpha$ ).

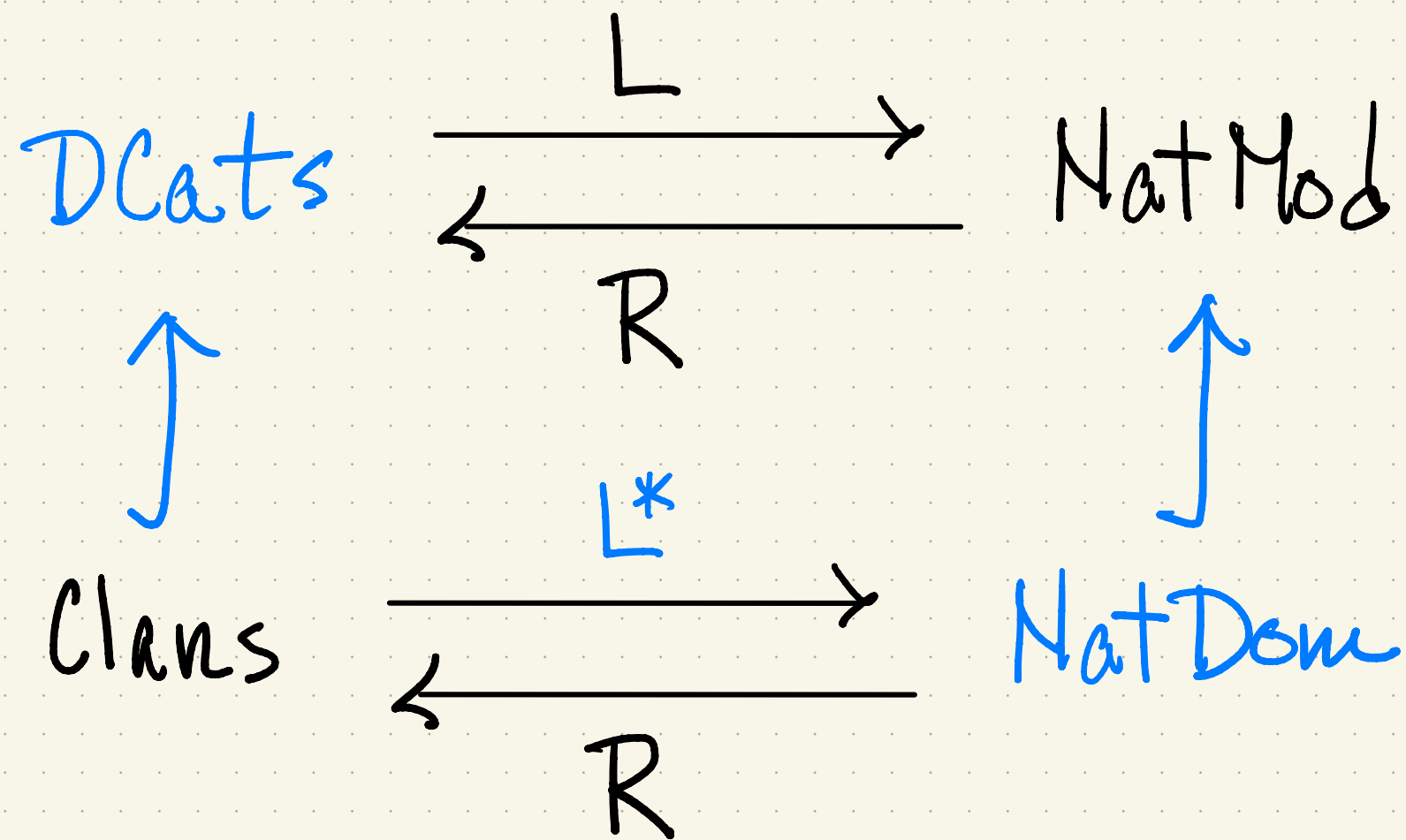
Fun Corollary the Soc  $t: 1 \rightarrow \Omega$  in  $\hat{\mathcal{P}}$   
is a MC-algebra.

Because it's the nerve of  $T: 1 \rightarrow 2$ ,

$$\begin{array}{ccc} 1 & & \vee 1 \\ t \downarrow & = & \downarrow \vee^T \\ \Omega & & \vee 2 \end{array}$$

\*)

In more detail :



$$L(\mathbb{C}, \mathcal{D}) = \perp\!\!\!\perp y^{\mathcal{D}}$$

$$R(\mathbb{C}, t) = (\mathbb{C}, \mathcal{D}_t)$$

$$L^* = \dots$$

- Given a natural model  $t: \hat{T} \rightarrow T$   
we freely add a monoid structure

$$\begin{array}{c}
 t \\
 \downarrow \\
 1 \longrightarrow u \longleftarrow u \cdot u
 \end{array}$$

- This is done by solving the domain equation

$$u \cong 1 + t \cdot u$$

- The solution is the colimit of the sequence

$$0 \rightarrow A_t 0 \rightarrow A_t^2 0 \rightarrow \dots$$

for the endofunctor  $A_t X = 1 + t \cdot X$  on  $\text{Poly}(\hat{G})$ .

- The colimit is  $t^* = 1 + t + t \cdot t + t^{\cdot 3} + \dots$   
 $= \sum_n t^{\cdot n}$ .

- So  $L^*(\mathbb{C}, \mathcal{D}) := L(\mathbb{C}, \mathcal{D})^*$ .

The proof uses 2 lemmas.

Lemma 1 If  $t: \hat{T} \rightarrow T$  is representable  
then the polynomial endofunctor

$$P_t: \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$$

has a right adjoint, and so preserves all colimits.

Lemma 2 If  $(\mathbb{C}, \mathcal{D})$  is a clan, then in

$$\begin{array}{ccc} (\mathbb{C}, \mathcal{D}) & \xrightarrow{\eta} & RL(\mathbb{C}, \mathcal{D}) \\ & \searrow^{\eta^*} & \downarrow \\ & & RL^*(\mathbb{C}, \mathcal{D}) \end{array}$$

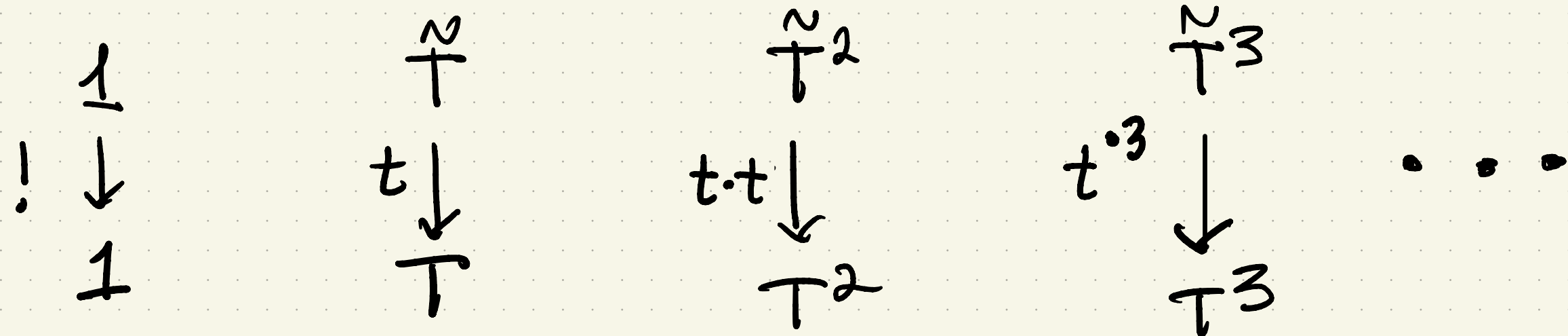
the unit  $\eta^*$  is an equivalence.



Note  $t^* = \Sigma t^n$  is the free completion of the

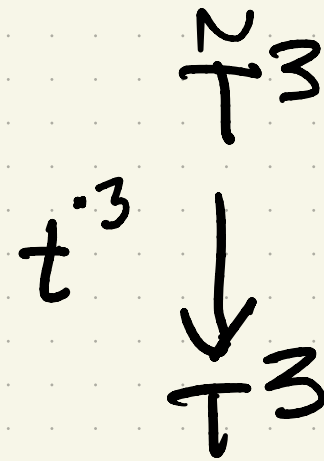
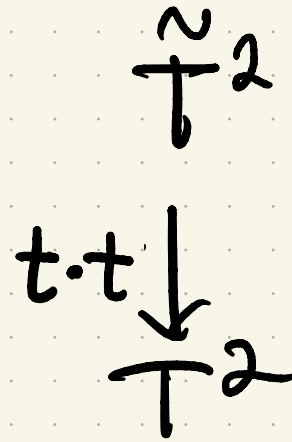
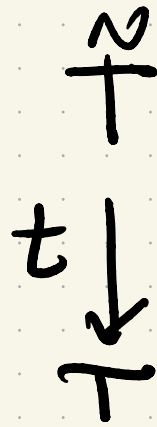
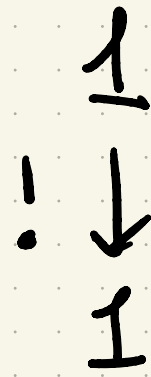
type theory  $t: \mathbb{T} \rightarrow T$  under  $\Sigma$ -types.

Consider the maps



as classifying types.

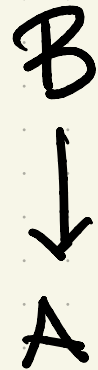
# Maps



...

# classify

A



...

# Contexts

A

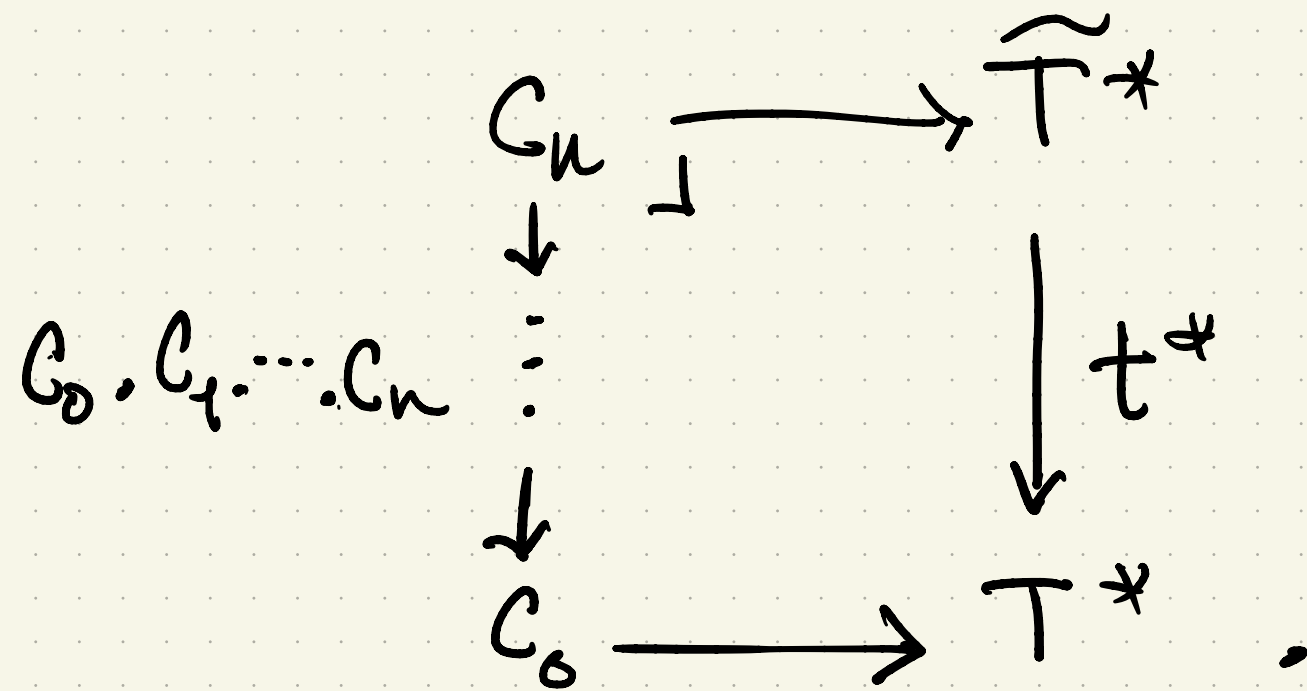
A.B

A.B.C

A.B.C.D

...

So  $t^*: \tilde{T}^* \rightarrow T^*$  classifies contexts of  $t$



the theory of contexts  $t^*$  of a theory  $t$

freely adds  $\Sigma$ -types.

THANKS!

