Algebraic Type Theory

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1. Natural Models

Def. A natural model consists of

- a category $\mathbb{C}$
- presheaves $T, \tilde{T}$
- a natural transformation
  $$t : \tilde{T} \to T$$
- which is representable

$$\forall c \in \mathbb{C} \quad \forall x \in T_c$$

$$\exists f : d \to c \quad \exists y \in \tilde{T}_d$$

$$\begin{tikzcd}
yd \arrow[r, "y"] \arrow[d, "yf"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{T} \arrow[d, "t"] \\
yc \arrow[r, "x"'] & T
\end{tikzcd}$$

This is equivalent to <u>CwF</u>:

- $\mathbb{C}$ cat of <u>contexts</u>
- T presheaf of <u>types</u>
- $\tilde{T}$ presheaf of <u>terms</u>
- <u>representability is context extension</u>.

$$\begin{tikzcd} & \tilde{T} \arrow[d, "t"] \\ yC \arrow[ur, dashed, "a"] \arrow[r, "A"'] & T \end{tikzcd}$$

$$C \vdash a : A$$

$$\begin{tikzcd} yC.A \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{T} \arrow[d, "t"] \\ yC \arrow[r, "A"'] & T \end{tikzcd}$$

- Type formers $1, \Sigma, \Pi$ are modelled by

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\quad 1", very near start] & \tilde{T} \arrow[d, "t"] \\ 1 \arrow[r] & T \end{tikzcd} \qquad \begin{tikzcd} \tilde{T}^2 \arrow[r] \arrow[d, "t^2"'] \arrow[dr, phantom, "\lrcorner\quad \Sigma", very near start] & \tilde{T} \arrow[d, "t"] \\ T^2 \arrow[r] & T \end{tikzcd} \qquad \begin{tikzcd} \tilde{T}_2 \arrow[r] \arrow[d, "t_2"'] \arrow[dr, phantom, "\lrcorner\quad \Pi", very near start] & \tilde{T} \arrow[d, "t"] \\ T_2 \arrow[r] & T \end{tikzcd}$$

- We will abstract this structure to form that of a

Martin-Löf algebra.

2. Polynomial Functors

Every $f : A \to B$ in an LCCC $\mathcal{E}$ determines a <u>polynomial endofunctor</u> $P_f : \mathcal{E} \to \mathcal{E}$.

$$
\begin{tikzcd}
\mathcal{E} \arrow[rrr, "Pf"] \arrow[dr, "A^*"'] & & & \mathcal{E} \\
& \mathcal{E}/A \arrow[r, "f_*"'] & \mathcal{E}/B \arrow[ur, "B!"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
X & X \times A \arrow[l] \arrow[d] & P_f X \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}
$$

- In the DTT of $\mathcal{E}$ $\quad P_f X = \sum_{b:B} X^{A_b}$.

- The UMP of $P_f X$ is

$$(b,x) : Z \longrightarrow P_f X$$

$$=\!=\!=\!=\!=\!=\!=\!=\!=\!=$$

$$\begin{tikzcd}
X & A_b \arrow[l, "x"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
 & Z \arrow[r, "b"'] & B
\end{tikzcd}$$
.

- The composite of polynomial functors is polynomial:

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "P_{g\cdot f}"] \arrow[dr, "P_f"'] & & \mathcal{E} \\ & \mathcal{E} \arrow[ur, "P_g"'] & \end{tikzcd}$$

$$\begin{array}{ccccc} A & C & & E \\ f\downarrow & g\downarrow & \rightsquigarrow & \downarrow g\cdot f \\ B & D & & F \end{array}$$

- As is $1_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$, so there is a monoid:

$$(\mathrm{Poly}\,\mathcal{E}\,,\ \cdot\,,\ 1_{\mathcal{E}})$$

3. M-L Algebras

Def A **M-L algebra** in a LCCC $\mathcal{E}$ is a map

$$t : \tilde{T} \longrightarrow T$$

with structure

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] \arrow[dr, phantom, "u"] & \tilde{T} \arrow[d, "t"] \\ 1 \arrow[r] & T \end{tikzcd} \qquad \begin{tikzcd} \tilde{T}^2 \arrow[r] \arrow[d, "t^2"'] \arrow[dr, phantom, "\lrcorner" very near start] \arrow[dr, phantom, "m"] & \tilde{T} \arrow[d, "t"] \\ T^2 \arrow[r] & T \end{tikzcd} \qquad \begin{tikzcd} \tilde{T}_2 \arrow[r] \arrow[d, "t_2"'] \arrow[dr, phantom, "\lrcorner" very near start] \arrow[dr, phantom, "c"] & \tilde{T} \arrow[d, "t"] \\ T_2 \arrow[r] & T \end{tikzcd}$$

where $P_{t^2} = P_{t \circ t} = P_t \circ P_t$ and $t_2 = P_t(t)$.

- the unit determines a natural transformation

$$u : 1_{\mathcal{E}} \longrightarrow P_t$$

- The multiplication determines another one

$$m : P_t \circ P_t \longrightarrow P_t$$

- the closure determines an algebra structure

$$c : P_t(t) \longrightarrow t$$

Dominance

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4. Examples

(i) A CwF $(\mathbb{C}, t : \tilde{T} \to T)$ has $1, \Sigma, \Pi$ as a CwF iff it is a ML-algebra in $\hat{\mathbb{C}}$.

(ii) Let $t : \tilde{T} \longrightarrow T$ be a ML-algebra in a LCCC $\mathcal{E}$.

Define a CwF $\hat{t} : Tm \to Ty$ by mapping in,

$$
\begin{array}{ccc}
Tm & & \mathcal{E}(-, \tilde{T}) \\
{\scriptstyle \hat{t}} \downarrow & := & \downarrow {\scriptstyle \mathcal{E}(-, t)} \\
Ty & & \mathcal{E}(-, T)
\end{array}
$$

Then $\hat{t}$ has $1, \Sigma, \Pi$ as a CwF.

Pf: Yoneda preserves ML-algebras, then use (i).

(iii) Clans

Let $t : \tilde{T} \to T$ be a natural model in $\hat{\mathbb{C}}$ and define <u>display maps</u> $\mathcal{D}_t$ in $\mathbb{C}$ by:

$$\mathcal{D}_t \ni f \Big\downarrow^{d}_{c} \quad \Leftrightarrow \quad \begin{tikzcd} yd \arrow[r] \arrow[d, "yf"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{T} \arrow[d, "t"] \\ yc \arrow[r] & T \end{tikzcd}$$

Then $\mathcal{D}_t$ is closed under pullbacks and under

- isos & composition if t is a dominance,
- pushforwards if t is closed.

So $(\mathbb{C}, \mathcal{D}_t)$ is a Π-clan if t is a ML-algebra.

Conversely given a Π-clan $(\mathbb{C}, \mathcal{D})$ there's a natural model $d : \tilde{D} \to D$ in $\widehat{\mathbb{C}}$ that's a ML-algebra, and $\mathcal{D} = \mathcal{D}_d$.

Pf Let

$$\begin{array}{c} \tilde{D} \\ d \big\downarrow \\ D \end{array} := \coprod_{f \in \mathcal{D}} yf \qquad \begin{array}{c} \coprod y\,\mathrm{dom} f \\ \big\downarrow \\ \coprod y\,\mathrm{cod} f \end{array}.$$

In fact, there's an adjunction*

$$\mathrm{Clans} \underset{R}{\overset{L}{\rightleftarrows}} \mathrm{NatMod}$$

where

$$L(\mathbb{C}, \mathfrak{D}) = \coprod y\mathfrak{D}$$

$$R(\mathbb{C}, t) = (\mathbb{C}, \mathfrak{D}_t) \quad .$$

(iv) Finite sets

In $\mathcal{E} = \mathrm{Set}$ let

$$\begin{array}{c}\widetilde{\mathbb{N}}\\ \mathrm{nat}\big\downarrow\\ \mathbb{N}\end{array} \;=\; \begin{array}{c}\sum_n [n]\\ \big\downarrow\\ \mathbb{N}\end{array} \;=\; \begin{tikzcd} \{m<n\} \arrow[d] & \subseteq \mathbb{N}\times\mathbb{N} \arrow[dl, "p_1"] \\ \mathbb{N} & \end{tikzcd}$$

The polynomial functor $P_{nat} : \mathrm{Set} \to \mathrm{Set}$ is then

$$P_{nat}(X) = \sum_n X^n = 1 + X + X^2 + \ldots \; .$$

• unit $\qquad u_X : X \longrightarrow 1 + X + \cdots$ $\qquad$ +-inclusion

• multiplication $\qquad m : P_{nat}^2 \longrightarrow P_{nat}$

$$\begin{tikzcd}
P_2 \arrow[r, "\tilde{m}"] \arrow[d, "nat^2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{N} \arrow[d] \\
\sum_n N^n = P_{nat} N \arrow[r, "m"'] & N
\end{tikzcd}$$

$$m_n : N^n \longrightarrow N$$

$$(k_1, \ldots, k_n) \longmapsto k_1 + \cdots + k_n$$

- unit $\quad u_X : X \longrightarrow 1 + X + \cdots$ $\quad$ +-inclusion

- multiplication $\quad m : P_{nat}^2 \longrightarrow P_{nat}$

$$\begin{tikzcd}
P_2 \arrow[r, "\tilde{m}"] \arrow[d, "nat^2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{\mathbb{N}} \arrow[d] \\
\sum_n \mathbb{N}^n = P_{nat}\mathbb{N} \arrow[r, "m"'] & \mathbb{N}
\end{tikzcd}$$

$$m_n : \mathbb{N}^n \longrightarrow \mathbb{N}$$

$$(k_1, \dots, k_n) \longmapsto k_1 + \dots + k_n$$

- closure : exercize !

(v) Bool

$$\begin{tikzcd} \tilde{2} \arrow[d, "T"'] \\ 2 \end{tikzcd} = \begin{tikzcd} \sum_{\delta:2} \delta \arrow[d] \\ 2 \end{tikzcd} = \begin{tikzcd} \{0\} \arrow[d] \\ \{0,1\} \end{tikzcd}$$

- $P_T(X) = \sum_{\delta:2} X^{\delta} = 1 + X$

- unit $\quad X \longrightarrow 1 + X \quad$ +-inclusion
- mult $\quad 1+(1+X) \longrightarrow 1+X \quad \nabla$
- closure

$$\begin{tikzcd} 1+1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\dashv"] & 1 \arrow[d] \\ 1+2 \arrow[r] & 2 \end{tikzcd} \qquad \text{SOC}$$

(vi) Groth. Universe

Take any cardinal α & do "the same thing":

$$\begin{array}{ll} \tilde{S}_\alpha & \text{“} \sum_{a \in S_\alpha} a \text{”} \\ \downarrow & \\ S_\alpha & \text{“sets of size} < \alpha \text{”} \end{array}$$

ML-algebra if α is inaccessible.

(vii) Syntactic CwF of DTT w/ $1, \Sigma, \Pi$

In $\widehat{Ctx}$:

$$\begin{array}{l} \mathrm{Terms}(-) = \{ - \vdash a : A \} \\ \quad \downarrow \\ \mathrm{Types}(-) = \{ - \vdash A \ \mathrm{type} \} \end{array}$$

the initial ML-algebra!

(viii) Hofmann-Streicher Universe

Given a cardinal $\alpha \in \mathrm{Set}$, for any $\mathbb{C}$, we have the α-H-S universe in $\hat{\mathbb{C}}$:

CwF

$$\mathcal{U} : \mathbb{C}^{op} \longrightarrow \mathrm{Set}_\alpha \qquad\qquad \tilde{\mathcal{U}}_\alpha$$

$$\cong \qquad \downarrow$$

$$E : \textstyle\int_{\mathbb{C}} \mathcal{U}^{op} \longrightarrow \mathrm{Set}_\alpha \qquad\qquad \mathcal{U}_\alpha$$

Nat Mod

Prop. $\tilde{\mathcal{U}}_\alpha \longrightarrow \mathcal{U}_\alpha$ is a ML-algebra (for suitable α).

This follows from 3 facts ...

Fact 1 $\tilde{\mathcal{U}}_\alpha \to \mathcal{U}_\alpha$ is the nerve of the universal α-small discrete fibration $\mathrm{Set}_\alpha^{\bullet\,op} \longrightarrow \mathrm{Set}_\alpha^{op}$ in Cat:

$$
\begin{tikzcd}
\hat{\mathbb{C}} \arrow[d, bend right, "\int_{\mathbb{C}}"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{Cat} \arrow[u, bend right, "\nu_{\mathbb{C}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{\mathcal{U}}_\alpha \arrow[d] \arrow[r, phantom, "="] & \nu_{\mathbb{C}}\mathrm{Set}_\alpha^{\bullet\,op} \arrow[d] \\
\mathcal{U}_\alpha & \nu_{\mathbb{C}}\mathrm{Set}_\alpha^{op}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\int_{\mathbb{C}} D \cong \mathbb{D} \arrow[r] \arrow[d, "\cdot"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Set}_\alpha^{\bullet\,op} \arrow[d] \\
\mathbb{C} \arrow[r, "D"'] & \mathrm{Set}_\alpha^{op}
\end{tikzcd}
$$

Fact 2 The nerve $N_{\mathbb{C}} : Cat \longrightarrow \hat{\mathbb{C}}$ preserves ML-algebras.

Fact 3 $Set_\alpha^{op} \longrightarrow Set_\alpha^{op}$ is a ML-algebra in Cat *
(for suitable α).

So indeed:

Prop. $\tilde{\mathcal{U}}_\alpha \longrightarrow \mathcal{U}_\alpha$ is a ML-algebra in $\hat{\mathbb{C}}$
(for suitable α).

Fun Corollary the SOC $t : 1 \longrightarrow \Omega$ in $\hat{\mathbb{C}}$ is a ML algebra.

Because it's the nerve of $T : 1 \longrightarrow 2$,

$$\begin{array}{ccc} 1 & & \nu 1 \\ t \big\downarrow & = & \big\downarrow \nu T \\ \Omega & & \nu 2 \end{array} .$$

*)

In more detail:

$$
\begin{tikzcd}
\mathrm{DCats} \arrow[r, shift left, "L"] & \mathrm{NatMod} \arrow[l, shift left, "R"] \\
\mathrm{Clans} \arrow[u, hook] \arrow[r, shift left, "L^*"] & \mathrm{NatDom} \arrow[u, hook] \arrow[l, shift left, "R"]
\end{tikzcd}
$$

$$L(\mathbb{C}, \mathfrak{D}) = \coprod y\mathfrak{D}$$

$$R(\mathbb{C}, t) = (\mathbb{C}, \mathfrak{D}_t)$$

$$L^* = \ldots$$

- Given a natural model $t : \tilde{T} \to T$ we freely add a monoid structure

$$\begin{tikzcd} & t \arrow[d] & \\ 1 \arrow[r] & u & u \cdot u \arrow[l] \end{tikzcd}$$

- This is done by solving the domain equation

$$u \cong 1 + t \cdot u .$$

- The solution is the colimit of the sequence

$$0 \to A_t 0 \to A_t^2 0 \to \cdots$$

for the endofunctor $A_t X = 1 + t \cdot x$ on $\mathrm{Poly}(\hat{\mathbb{C}})$.

- The colimit is $t^* = 1 + t + t \cdot t + t^{\cdot 3} + \cdots$

$$= \sum_n t^{\cdot n} .$$

- So $L^*(\mathbb{C}, \mathbb{D}) := L(\mathbb{C}, \mathbb{D})^*$.

The proof uses 2 lemmas.

Lemma 1 If $t : \tilde{T} \longrightarrow T$ is representable

then the polynomial endofunctor

$$P_t : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$$

has a right adjoint, and so preserves all colimits.

Lemma 2 If $(\mathbb{C}, \mathcal{D})$ is a clan, then in

$$\begin{tikzcd} (\mathbb{C},\mathcal{D}) \arrow[r, "\eta"] \arrow[dr, "\sim", "\eta^*"'] & RL(\mathbb{C},\mathcal{D}) \arrow[d] \\ & RL^*(\mathbb{C},\mathcal{D}) \end{tikzcd}$$

the unit η^* is an equivalence.

Note $t^* = \Sigma t^{\cdot n}$ is the free completion of the type theory $t : \tilde{T} \longrightarrow T$ under Σ-types.

Consider the maps

$$
\begin{array}{ccccc}
1 & \tilde{T} & \tilde{T}^2 & \tilde{T}^3 & \\
! \downarrow & t \downarrow & t \cdot t \downarrow & t^{\cdot 3} \downarrow & \cdots \\
1 & T & T^2 & T^3 &
\end{array}
$$

as classifying types.

Maps

$$
\begin{array}{ccccc}
1 & \tilde{T} & \tilde{T}^2 & \tilde{T}^3 & \\
{\scriptstyle !}\downarrow & {\scriptstyle t}\downarrow & {\scriptstyle t\cdot t}\downarrow & {\scriptstyle t^{\cdot 3}}\downarrow & \cdots \\
1 & T & T^2 & T^3 &
\end{array}
$$

classify

$$
\begin{array}{ccccc}
 & & & D & \\
 & & C & \downarrow & \\
 & B & \downarrow & C & \\
A & \downarrow & B & \downarrow & \cdots \\
 & A & \downarrow & B & \\
 & & A & \downarrow & \\
 & & & A &
\end{array}
$$

Contexts

$A \qquad A.B \qquad A.B.C \qquad A.B.C.D \qquad \cdots$

So $t^*: \widetilde{T}^* \to T^*$ classifies <u>contexts of t</u>

$$\begin{tikzcd}
C_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{T}^* \arrow[dd, "t^*"] \\
\vdots \arrow[d] & \\
C_0 \arrow[r] & T^*
\end{tikzcd}$$

$C_0 . C_1 . \cdots . C_n$

the <u>theory of contexts</u> t^* of a theory t

freely adds Σ-types.

THANKS!