Dialectica Categories and Doctrines

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Workshop on Doctrines and Fibrations

(joint work with Davide Trotta and Matteo Spadetto)

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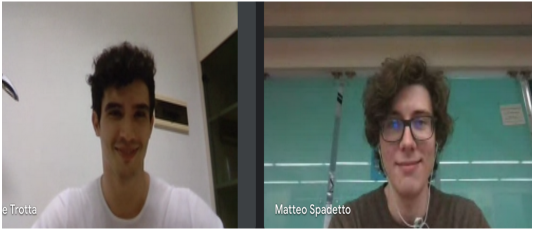
Thanks!

Milly Maietti for the invitation today!

Davide for the first invitation to discuss this, some three years ago.

Davide and Matteo for all the work*!



* Dialectica Principles via Gödel Doctrines, TCS 2023.
Dialectica logical principles: not only rules, JLC 2022.

Personal stories

Elegant mathematics will of itself tell a tale, and one with the merit of simplicity. This may carry philosophical weight. But that cannot be guaranteed: in the end one cannot escape the need to form a judgement of significance.
Martin Hyland, 2004

Personal stories

Personal stories

Dialectica Interpretation

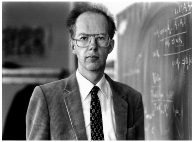
Dialectica Interpretation (Gödel 1958): an interpretation of intuitionistic arithmetic HA in a quantifier-free theory of functionals of finite type **System $\mathbf{T}$**.

Idea: translate every formula A of HA to

$$A^D = \exists u \forall x A_D$$

where A_D is quantifier-free.

Dialectica Interpretation



Application (Gödel 1958): if HA proves A, then System T proves $A_D(t, x)$, where x is a string of variables for functionals of finite type, and t a suitable sequence of terms (not containing x).

Goal: to be as **constructive** as possible, while being able to interpret all of classical Peano arithmetic (Troelstra).

Gödel (1958), *Über eine bisher noch nicht benützte erweiterung des finiten standpunktes.*, Dialectica, 12(3-4):280–287. (Translation in Gödel's Collected Works)

Dialectica interpretation

$A_D(u;x)$ quantifier-free formula defined inductively:

$$
\begin{array}{lcl}
(P)_D & \equiv & P \ (P \text{ atomic}) \\
(A \wedge B)_D(u,v;x,y) & \equiv & A_D(u;x) \wedge B_D(v;y) \\
(A \vee B)_D(u,v,z;x,y) & \equiv & (z = 0 \to A_D(u;x)) \wedge (z \neq 0 \to B_D(v;y)) \\
(A \to B)_D(f,F;u,y) & \equiv & A_D(u;Fuy) \to B_D(fu;y) \\
(\exists z A)_D(u,x;z) & \equiv & A_D(u;x) \\
(\forall z A)_D(f;y,z) & \equiv & A_D(fz;y)
\end{array}
$$

Theorem (Dialectica Soundness, Gödel 1958)

Whenever a formula A is provable in Heyting arithmetic then there exists a sequence of closed terms t such that $A_D(t;y)$ is provable in system T. The sequence of terms t and the proof of $A_D(t;y)$ are constructed from the given proof of A in Heyting arithmetic.

Dialectica interpretation

The most complicated clause of the translation is the definition of the translation of the **implication connective** $(A \to B)^D$

$$(A \to B)^D = \exists f, F \forall u, y (A_D(u, F(u, y)) \to B_D(f(u), y)).$$

Intuition: Given a witness u in U for the hypothesis A_D, there exists a function f assigning a witness $f(u)$ to B_D. Moreover, from a counterexample y to the conclusion B_D, we should be able to find a counterexample $F(u, y)$ for the hypothesis A_D.

Feferman et al editors (1986), *Kurt Gödel: Collected Works: Volume II*, Oxford University Press.

Dialectica interpretation

Troelstra (p 226 Collected Works Gödel) from Spector (1962)

$$[\exists u \forall x . A_D(u,x) \to \exists v \forall y . B_D(v,y)] \leftrightarrow^{(i)}$$

$$[\forall u (\forall x A_D(u,x) \to \exists v . \forall y (B_D(v,y))] \leftrightarrow^{(ii)}$$

$$[\forall u \exists v (\forall x . A_D(u,x) \to \forall y B_D(v,y))] \leftrightarrow^{(iii)}$$

$$[\forall u \exists v \forall y (\forall x A_D(u,x) \to B_D(v,y))] \leftrightarrow^{(iv)}$$

$$[\forall u \exists v \forall y \exists x (A_D(u,x) \to B_D(v,y))] \leftrightarrow^{(v)}$$

$$\exists V, X \forall u, y (A_D(u, X(u,y)) \to B_D(V(u), y))$$

where (i) and (iii) are intuitionistic, but (ii) requires **Independence of Premise**, (iv) requires **Markov Principle** and (v) requires two uses of the **axiom of choice**.

Dialectica interpretation

Hence translation involves three logical, non-intuitionistic, principles:

1. **Principle of Independence of Premise** (IP)

$$(A \to \exists v.B(v)) \to \exists v.(A \to B(v))$$

2. a generalisation/modification of **Markov Principle** (MMP)

$$(\forall x.A(x) \to B(y)) \to \exists x.(A(x) \to B(y))$$

3. the **axiom of choice** (AC)

$$\forall y.\exists x.A(x,y) \to \exists V.\forall y.A(V(y),y)$$

Categorical Dialectica Construction

Dialectica category (de Paiva 1988): Given a category C with finite limits, one can build a new category $\mathfrak{Dial}(C)$, whose objects have the form $A = (U, X, \alpha)$ where α is a subobject of $U \times X$ in C; **think** of this object as representing the formula

$$\exists u \forall x \alpha(u, x).$$

A map from $\exists u \forall x \alpha(u, x)$ to $\exists v \forall y \beta(v, y)$ can be thought of as a pair $(f : U \to V, F : U \times Y \to X)$ of terms/maps, subject to the entailment condition

$$\alpha(u, F(u, y)) \vdash \beta(f(u), y).$$

(First internalisation of the Dialectica interpretation!)

Original Dialectica Constructions

Thesis: 4 chapters, 4 main theorems.
All of them of the form:
Category C is a categorical model of logic *L*.

all start from C cartesian closed cat + coproducts + (...)

Thm 1: $\mathfrak{Dial}$(C) is a model of !-free ILL

Thm 2: $\mathfrak{Dial}$(C) + ! (where ! is a co-free monoidal comonad) is a model of IL

Thm 3: Gir(C) (a simpler dialectica cat) is a model of (!,?)-free CLL/FILL

Thm 4: Gir(C) + !,? (!,? given by a composite monoidal (co)monad) is a model of IL/CL

This Talk: only the first half

Only 2 main theorems:

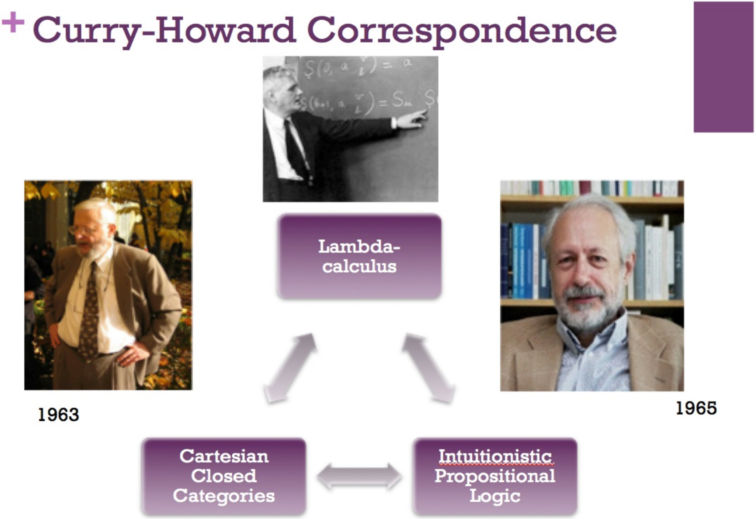
Start with C a cartesian closed cat + coproducts + (...)

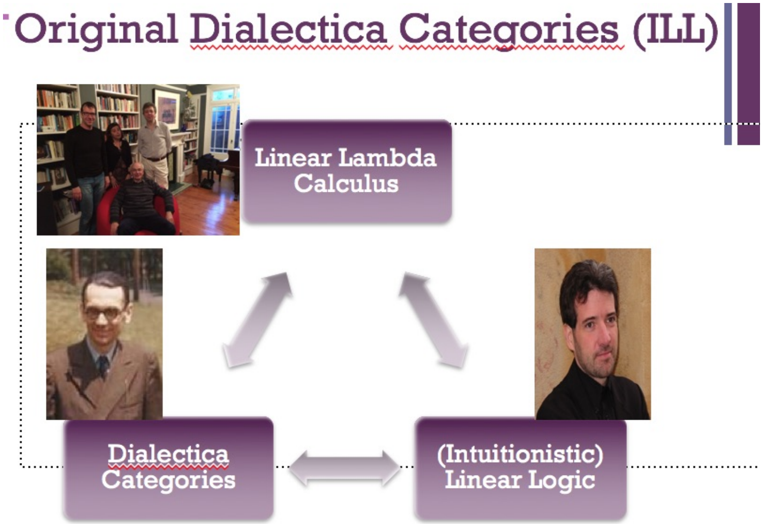
Apply Dialectica construction to it get to $\mathfrak{Dial}(C)$

Thm 1: $\mathfrak{Dial}(C)$ is a model of !-free Intuitionistic Linear Logic

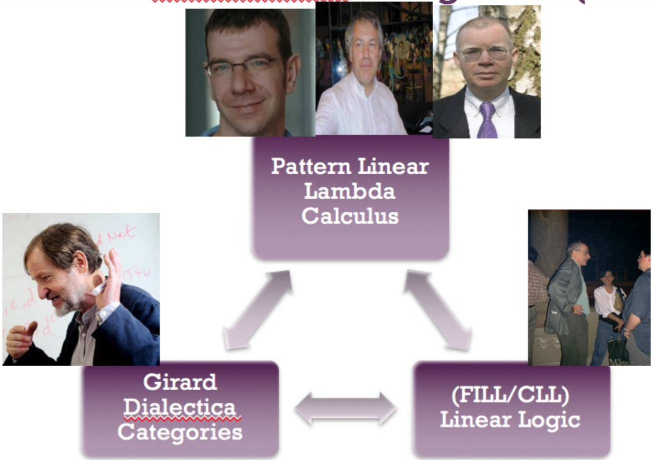
Thm 2: $\mathfrak{Dial}(C)$ + !, where ! is a co-free monoidal comonad, is a model of $IL_{\to,\wedge}$ or simply typed lambda-calculus

Why this is interesting?


+ Curry-Howard Correspondence
Lambda-calculus
1963
1965
Cartesian Closed Categories
Intuitionistic Propositional Logic


Original Dialectica Categories (ILL)
Linear Lambda Calculus
Dialectica Categories
(Intuitionistic) Linear Logic

Girard Dialectica categories (CLL)



Challenges of modeling Linear Logic

Traditional categorical modeling of intuitionistic logic:

- formula $A \rightsquigarrow$ object A of appropriate category
- $A \wedge B \rightsquigarrow A \times B$ (real product)
- $A \to B \rightsquigarrow B^A$ (set of functions from A to B)
- These are real products, so we have projections ($A \times B \to A, B$) and diagonals ($A \to A \times A$) which correspond to deletion and duplication of resources

Not Linear!!! New in the late 80's, now perhaps the baseline, we want a monoidal structure.
Propositional Easy: Need to use *tensor products* and *internal homs* in CT $\Rightarrow$ symmetric monoidal closed category, Kelly (in the 60s)

Hard: how to define the *make-everything-usual* operator "!"

Category $\mathfrak{Dial}(C)$

Start with a cat C that is cartesian closed with pullbacks. Then build a new category $\mathfrak{Dial}(C)$.

Objects are relations in C, triples (U, X, α), $\alpha : U \times X \to 2$, so either $u\alpha x$ or not.

Maps are pairs of maps in C. A map from $A = (U, X, \alpha)$ to $B = (V, Y, \beta)$ is a pair of maps in C, $(f : U \to V, F : U \times Y \to X)$ such that a 'semi-adjunction condition' is satisfied: for $u \in U, y \in Y$, $u\alpha F(u, y)$ implies $fu\beta y$. (**Note direction and dependence!**)

Theorem1: (de Paiva 1987) [Linear structure]

The category $\mathfrak{Dial}$(C) has a symmetric monoidal closed structure (and products, **weak** coproducts), that makes it a model of (exponential-free) **intuitionistic** linear logic.

Intuition for these objects?

Blass makes the case for thinking of problems in terms of computational complexity. Samuel da Silva and I say you can think of **Kolmogorov-Veloso problems** $\Rightarrow$ applications to Set Theory. Many other interpretations make sense.
Intuitively an object of $\mathfrak{Dial}(C)$

$$A = (U, X, \alpha)$$

can be seen as representing a problem.
The elements of U are instances of the problem, while the elements of X are possible answers to the problem instances.
The relation α checks whether the answer is correct for that instance of the problem or not.

(Superpower games?)

Examples of objects in $\mathfrak{Dial}(C)$

1. The object $(\mathbb{N}, \mathbb{N}, =)$ where n is related to m iff $n = m$.
2. The object $(\mathbb{N}^{\mathbb{N}}, \mathbb{N}, \alpha)$ where f is α-related to n iff $f(n) = n$.
3. The object $(\mathbb{R}, \mathbb{R}, \leq)$ where r_1 and r_2 are related iff $r_1 \leq r_2$
4. The objects $(2, 2, =)$ and $(2, 2, \neq)$ with usual equality inequality.

The point of Dialectica categories?

A model of Linear Logic, instead of Intuitionistic Logic. (Justifies LL in terms of a traditional proof-theoretic tool and conversely explains the traditional tool in terms of a ‘modern’ linear, resource conscious decomposition.)

A *good* model of Linear Logic: keep the differences that Girard wanted to make. (work with Andrea Schalk on L-valued models of LL).

Justifies claims about Curry-Howard and Harper’s Trinitarism, connections to programming and using CT as syntax guidance.

Now: work with Trotta and Spadetto allows us to see where the assumptions in Gödel’s argument (hacks?) are used, based on Hofstra’s explanation.

More Categorical Dialectica Constructions

Work in the original Dialectica categories (de Paiva 1989, 1991) on the categorical structure needed to model Linear Logic (Girard 1987).

Model is pretty cool! Lots of recent work on it, 30+ years later.
Generalization: initial construction has been generalized for arbitrary fibrations, by Hyland, Biering, Hofstra, von Glehn, Moss, etc.
Trotta, Spadetto and V. describe a categorical version of Dialectica in terms of (Lawvere's) **doctrines**.
But **why** do we do it? Isn't the modelling using categories enough?

Hofstra (2011), *The dialectica monad and its cousins.*, A tribute to M. Makkai
Trotta, Spadetto and de Paiva (2021), *The Gödel fibration.*, MFCS 2021
Trotta, Spadetto and de Paiva (2022), *Gödel Doctrines.*, LFCS 2022

Dialectica via Doctrines

Two reasons:

1. First-order is of course more expressive than propositional logic, sometimes we need the extra expressivity;
2. Much tighter correspondence between the logic and the category theory, as exemplified by the Dialectica logical principles paper

In particular we get the ability to show how the internalisation of morphisms work.

Dialectica via Doctrines

How well does the construction of the Dialectica categories (or doctrines) capture the essential ingredients of Gödel's original interpretation?

1. Given a doctrine P, when is there a doctrine P' such that $\mathfrak{Dial}(P') \cong P$?

2. When such doctrine P' exists, how do we find it?

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2. When such doctrine P' exists, how do we find it? P' is given by the **quantifier-free elements** of the Gödel doctrine P

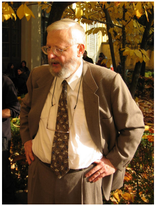
Dialectica via Doctrines

As we saw the Dialectica translation requires some classical principles:
independence of premise(IP)
Markov principle (MP)
and the axiom of choice (AC).

How can we see these principles in our categorical modelling?

Can these categories and these principles be described in more conceptual terms, for example, in terms of universal properties?

Doctrines



Lawvere defined hyperdoctrines, we start with less.

Definition

A **doctrine** is just a functor from a category $\mathcal{C}$ with finite products, to Pos, the category of posets.

$$P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$$

Existential and Universal Doctrines

Definition (existential/universal doctrines)

A doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ is ***existential*** (resp. ***universal***) if, for every A_1 and A_2 in $\mathcal{C}$ and every projection $A_1 \times A_2 \xrightarrow{\pi_i} A_i$, $i = 1, 2$, the functor:

$$PA_i \xrightarrow{P_{\pi_i}} P(A_1 \times A_2)$$

has a left adjoint $\exists_{\pi_i}$ (resp. a right adjoint $\forall_{\pi_i}$), and these satisfy the ***Beck-Chevalley conditions***.

(Davide Trotta (PhD work TAC 2020): "The existential completion"exists and satisfies all 2-categorical properties you may want. Ditto for the universal completion.)

Doctrines and quantifier-free formulas

We want a suitable universal property to represent predicates that are quantifier-free, **categorically**. We have dual definitions for existential and universal quantifiers.

The paper defines:

existential splitting predicates, (use internal language, a weak universal property) – Maietti/Trotta and Frey 2021.

[$\alpha(a)$ in $P(A)$ is **existential splitting** (ES) if for all $\beta(a,b)$ such that $\alpha(a) \to \exists b.\beta(a,b)$ there is a $g : A \to B$ such that $\alpha(a) \to \beta(a, g(a))$]

existential-free predicates (ES predicates stable under reindexing),

doctrines P with enough existential-free predicates.

as well as their duals

Trotta, Spadetto and de Paiva (2023), *Dialectica principles via Gödel doctrines*, TCS 2023

Definition (Gödel doctrine)

A doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ is called a ***Gödel doctrine*** if:

1. the category $\mathcal{C}$ is cartesian closed;
2. the doctrine P is existential and universal;
3. the doctrine P has enough existential-free predicates;
4. the existential-free objects of P are stable under universal quantification, i.e. if $\alpha \in P(A)$ is existential-free, then $\forall_\pi(\alpha)$ is existential-free for every projection π from A;
5. the sub-doctrine $P'\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ of the existential-free predicates of P has enough universal-free predicates.

a mouthful! without item 5 we call it a ***Skolem doctrine***.

Definition (Dialectica doctrine, after Hofstra 2011)

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ be a doctrine whose base category C is cartesian closed. The **dialectica doctrine** $\mathfrak{Dial}(P)\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ is defined as the functor sending an object I into the poset $\mathfrak{Dial}(P)(I)$ defined as follows:

objects are 4-tuples (I, U, X, α) where I, X and U are objects of the base category $\mathcal{C}$ and $\alpha \in P(I \times U \times X)$;

partial order: we say that $(I, U, X, \alpha) \leq (I, V, Y, \beta)$ if there exists a pair (f_0, f_1), where $I \times U \xrightarrow{f_0} V$ and $I \times U \times Y \xrightarrow{f_1} X$ are morphisms of $\mathcal{C}$ such that:

$$\alpha(i, u, f_1(i, u, y)) \leq \beta(i, f_0(i, u), y).$$

This is a direct adaptation to the proof irrelevant setting of Hofstra's definition of Dialectica fibration.

Theorem (Hofstra 2011)

If $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ *is a doctrine, then there is an isomorphism* $\mathfrak{Dial}(P) \cong (P^{\forall})^{\exists}$ *which is natural in* P.

(Here $Q^{\forall}$ and $Q^{\exists}$ denote the universal and the existential completions of any doctrine Q.)

Theorem (Trotta, Spadetto, dP2022)

Every Gödel doctrine P *is equivalent to the Dialectica completion* $\mathfrak{Dial}(P')$ *of the full subdoctrine* P' *of* P *consisting of the quantifier-free predicates of* P.

Gödel doctrines in action

Five theorems show the modelling provided by Gödel doctrines is very tight.

1. For a Gödel doctrine P and any predicate α of $P(A)$, there exists a quantifier-free predicate α_D of $P(I \times U \times X)$ such that:

$$i : I \mid \alpha(i) \dashv\vdash \exists u : U . \forall x : X . \alpha_D(i, u, x).$$

Thus in a Gödel doctrine every formula admits a presentation of the exact form used in the Dialectica interpretation.

2. Morphisms of the dialectica categories correspond to implication in the Gödel doctrines.

3. Skolemisation in Dialectica is modelled by Gödel doctrines, actually need Gödel hyperdoctrines. A hyperdoctrine $P \colon \mathcal{C}^{op} \longrightarrow$ Hey is called a ***Gödel hyperdoctrine*** when P is a Gödel doctrine.

Gödel doctrines in action

4. Every Gödel hyperdoctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ satisfies the ***Rule of Independence of Premise***

5. Every Gödel hyperdoctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ satisfies the following ***Modified Markov Rule***: whenever $\beta_D \in P(A)$ is a quantifier-free predicate and $\alpha \in P(A \times B)$ is an existential-free predicate, it is the case that:

$$a : A \mid \top \vdash (\forall b.\alpha(a,b)) \to \beta_D(a)$$

implies that

$$a : A \mid \top \vdash \exists b.(\alpha(a,b) \to \beta_D(a)).$$

Summarizing

Used existential and universal doctrines (and their completions) to provide notions of quantifier-free formulae

Showed that the Gödel doctrines satisfy:
Dialectica Normal Form
Soundness of Implication
Skolemisation
Independence of Premise
Markov Principle

Obtained a very faithful categorical description of the Dialectica interpretation.

Original models have several applications (games, set theory), functional and imperative programming, concurrency, automata theory. Doctrinal ones should have apps too! Generalized proof-relevant version CT2023 – J. Weinberger, D. Trotta, V.

Thank you!

Some References

(see https://github.com/vcvpaiva/DialecticaCategories)

- Trotta, Spadetto, and de Paiva, *Dialectica Principles via Gödel Doctrines.* In Theoretical Computer Science. (2023)
- Maietti and Trotta, *A characterization of generalized existential completions.* APAL 2023
- Frey, *Categories of partial equivalence relations as localizations*, JPAA 2022.
- P. Hofstra, *The Dialectica Monad and its cousins*, Proc in honour of M. Makkai, 2011.
- de Paiva, *The Dialectica Categories*, In Proc of Categories in Computer Science and Logic, Boulder, CO, 1987. Contemporary Mathematics, vol 92, American Mathematical Society, 1989 (eds. J. Gray and A. Scedrov)