Elementary fibrations and quotients of groupoids

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joint work with Fabio Pasquali and Pino Rosolini

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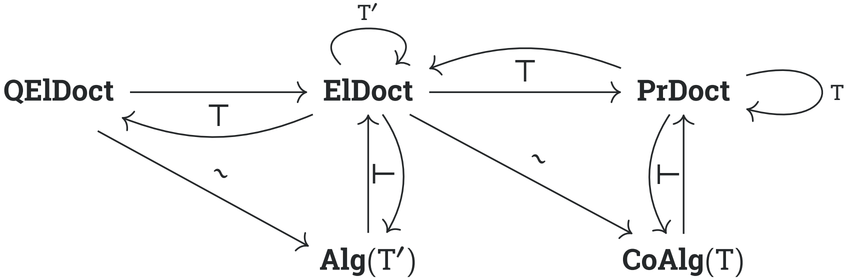
Introduction

In the theory of doctrines the tight link between equality and quotients is expressed by the following (co)monadicities:



where the 2-comonad T is colax-idempotent and it induces the 2-monad T′.

M.E. Maietti, G. Rosolini. Elementary quotient completion. *Theory Appl. Categ.* 27, 2013.

F. Pasquali. A co-free construction for elementary doctrines. *Appl. Categ. Structures* 23, 2015.

D. Trotta. Existential completion and pseudo-distributive laws. Ph.D. thesis, U. Trento, 2019.

J.E., F. Pasquali, G. Rosolini. Elementary doctrines as coalgebras. *J. Pure Appl. Algebra* 224, 2020.

Elementary fibrations I

A fibration $p\colon \mathcal{E} \twoheadrightarrow \mathcal{B}$ is elementary if

1. it has finite products,
2. for every Z, X in $\mathcal{B}$ and $A \in \mathcal{E}_{Z\times X}$, there is a cocartesian lift at A over $\mathrm{pr}_{1,2,2}$

$$\begin{array}{ccc} A & \xrightarrow{\delta^Z_A} & \exists_{Z,X}A \\ Z \times X & \xrightarrow{\mathrm{pr}_{1,2,2}} & Z \times X \times X \end{array}$$

3. and cocartesian arrows over the parametrised diagonals $\mathrm{pr}_{1,2,2}$ are product-stable and pairable.

$$\begin{array}{ccccccc} C' & \longleftarrow & C' \wedge A' & \longrightarrow & A' & \longrightarrow & A \\ \downarrow & & \downarrow \psi & & \downarrow \varphi' & & \downarrow \varphi \\ C & \longleftarrow & C \wedge B' & \longrightarrow & B' & \longrightarrow & B \end{array} \quad \xmapsto{\;p\;} \quad \begin{array}{ccc} Z \times X & \xrightarrow{f \times X} & V \times X \\ \downarrow \mathrm{pr}_{1,2,2} & & \downarrow \mathrm{pr}_{1,2,2} \\ Z \times X \times X & \xrightarrow[f \times X \times X]{} & V \times X \times X \end{array}$$

Elementary fibrations - Examples

1. The fibration of predicates over contexts for a first-order language with equality. More generally, elementary doctrines are (essentially) faithful elementary fibrations.
2. The fibrations $\mathrm{Sub}_{\mathcal{C}}$ and $\mathrm{Var}_{\mathcal{C}}$, when $\mathcal{C}$ has finite products and (weak) pullbacks.
3. The fibration $\mathrm{cod}\colon \mathcal{M} \longrightarrow\!\!\!\!\rightarrow \mathcal{C}$, when $(\mathcal{E}, \mathcal{M})$ is a (suitable) orthogonal factorisation system on $\mathcal{C}$.
4. The fibration $\mathrm{Fam}(\mathcal{C}) \longrightarrow\!\!\!\!\rightarrow \mathcal{S}et$, when $\mathcal{C}$ has finite products and a strict initial object.
5. The fibration $\mathrm{cod}\colon \mathcal{C}^2 \longrightarrow\!\!\!\!\rightarrow \mathcal{C}$, when $\mathcal{C}$ has finite limits.
6. $\mathrm{cod}\colon \mathrm{SCIsoFib} \longrightarrow\!\!\!\!\rightarrow \mathcal{C}at$, where $\mathrm{SCIsoFib} =$ split isofibrations and morphisms preserving the cleavage on the nose.

Elementary fibrations II

$p\colon \mathcal{E} \longrightarrow\!\!\!\!\rightarrow \mathcal{B}$ an elementary fibration.

The cocartesian lift at $A \in \mathcal{E}_{Z\times X}$ is determined by that at $\top_X$:

$$
\begin{array}{ccccccc}
A & \longleftarrow & A & \longrightarrow & \top_{Z\times X} & \longrightarrow & \top_X \\
\downarrow & & \Big\downarrow{\scriptstyle \delta^Z_A} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \delta_X} \\
\mathrm{pr}_{1,2}{}^*A & \longleftarrow & \exists_{Z,X}A & \longrightarrow & \mathrm{pr}_{2,3}{}^*\mathrm{I}_X & \longrightarrow & \mathrm{I}_X
\end{array}
\quad \overset{p}{\longmapsto} \quad
\begin{array}{ccc}
Z\times X & \xrightarrow{\mathrm{pr}_2} & X \\
\Big\downarrow{\scriptstyle \mathrm{pr}_{1,2,2}} & & \Big\downarrow{\scriptstyle \mathrm{pr}_{1,1}} \\
Z\times X\times X & \xrightarrow[\mathrm{pr}_{2,3}]{} & X\times X
\end{array}
$$

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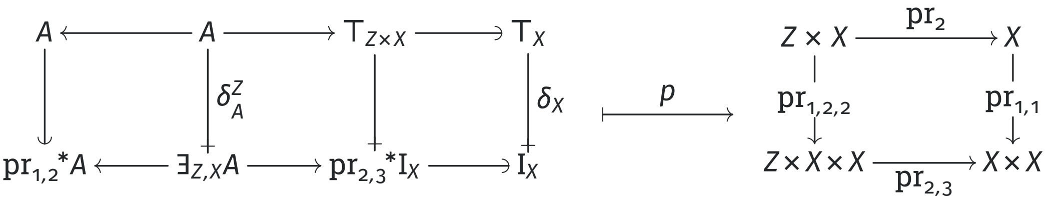


Then

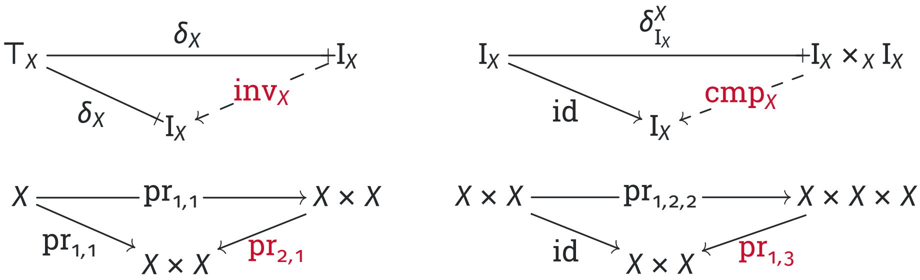
$$\exists_{Z,X}(A) \cong \mathrm{pr}_{1,2}{}^{*}A \wedge \mathrm{pr}_{2,3}{}^{*}\mathrm{I}_X = A \times_X \mathrm{I}_X$$

and WLOG

$$\delta^Z_A = \langle \mathrm{id}_A, \delta_X \circ !_A \rangle \colon A \to A \times_X \mathrm{I}_X$$

Groupoids from elementary fibrations

$p\colon \mathcal{E} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{B}$ an elementary fibration.
Then $\mathrm{I}_X \in \mathcal{E}_{X\times X}$ has a groupoid structure with unit the loop δ_X and



where $\mathrm{I}_X \times_X \mathrm{I}_X = {\mathrm{pr}_{1,2}}^* \mathrm{I}_X \wedge {\mathrm{pr}_{2,3}}^* \mathrm{I}_X$.

Groupoid equations hold since they hold on "reflexivities" $\delta_{(-)}$.

Groupoids in fibrations with products

$p\colon \mathcal{E} \twoheadrightarrow \mathcal{B}$ a fibration with products.

A *p-groupoid* $\mathbb{X}$ on $X \in \mathcal{B}$ is a groupoid in $\mathcal{E}$ on the fibred terminal object $\top_X$, sitting over the codiscrete groupoid on X:

$$\begin{array}{ccccc} \mathcal{E} & \qquad \top_X & \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} & \bar{X} & \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} & \bar{X} \times_X \bar{X} \\ \big\downarrow & & & & & \\ \mathcal{B} & \qquad X & \overset{\overset{\mathrm{pr}_1}{\longleftarrow}}{\underset{\underset{\mathrm{pr}_2}{\longleftarrow}}{\xrightarrow{\mathrm{pr}_{1,1}}}} & X \times X & \overset{\overset{\mathrm{pr}_{1,2}}{\longleftarrow}}{\underset{\underset{\mathrm{pr}_{2,3}}{\longleftarrow}}{\xleftarrow{\mathrm{pr}_{1,3}}}} & X \times X \times X \end{array}$$

p-groupoids form a category $\mathcal{Gpd}(p)$ fibred over $\mathcal{B}$ via $\mathrm{Ob}\colon \mathbb{X} \mapsto X$.

Examples:

1. If p is elementary, then $\mathrm{I}_X = (X, \mathrm{I}_X)$ is a p-groupoid.
2. If $\mathcal{C}$ has pullbacks and $p = \mathrm{cod}\colon \mathcal{C}^2 \twoheadrightarrow \mathcal{C}$, then $\mathcal{Gpd}(p) = \mathcal{Gpd}(\mathcal{C})$.
3. For $(\mathcal{V}, \otimes, I)$, if $p = \mathrm{Fam}(\mathcal{V}) \twoheadrightarrow \mathcal{Set}$, then $\mathcal{Gpd}(p) = \mathcal{V}\text{-}\mathcal{Gpd}$.

Actions of p-groupoids

An **action** of a p-groupoid $\mathbb{X} = (X, \bar{X}, \mathrm{un}_{\mathbb{X}}, \mathrm{cmp}_{\mathbb{X}}, \mathrm{inv}_{\mathbb{X}})$ is given by $A \in \mathcal{E}_X$ and

$$\begin{array}{ccc} A \times_X \bar{X} & \xrightarrow{\quad\alpha\quad} & A \\ X \times X & \xrightarrow{\quad\mathrm{pr}_2\quad} & X \end{array}$$

making two diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}} & A \\ {\scriptstyle \langle \mathrm{id}_A, \mathrm{un}_{\mathbb{X}} \circ !_A \rangle}\downarrow & & \| \\ A \times_X \bar{X} & \xrightarrow{\alpha} & A \end{array} \qquad \begin{array}{ccc} A \times_X \bar{X} \times_X \bar{X} & \xrightarrow{\alpha \times \mathrm{id}} & A \times_X \bar{X} \\ {\scriptstyle \mathrm{id} \times \mathrm{cmp}_{\mathbb{X}}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ A \times_X \bar{X} & \xrightarrow{\alpha} & A \end{array}$$

over $\quad \mathrm{pr}_2 \circ \mathrm{pr}_{1,1} = \mathrm{id} \quad$ and $\quad \mathrm{pr}_2 \circ \mathrm{pr}_{1,3} = \mathrm{pr}_2 \circ \mathrm{pr}_{2,3}.$

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over $\mathrm{pr}_2 \circ \mathrm{pr}_{1,1} = \mathrm{id}$ and $\mathrm{pr}_2 \circ \mathrm{pr}_{1,3} = \mathrm{pr}_2 \circ \mathrm{pr}_{2,3}$.

If p is elementary, $\mathcal{A}ct(\mathrm{I}_X) \cong \mathcal{E}_X$ since $\delta_A^Z = \langle \mathrm{id}_A, \delta_X \circ !_A \rangle : A \longrightarrow A \times_X \mathrm{I}_X$ is cocartesian.

The reader comonad

$\mathcal{B}$ a category with finite products.
The reader comonad $(-) \times X\colon \mathcal{B} \to \mathcal{B}$ induces a monad M_X on its Kleisli category

$$(-) \times X \circlearrowright \mathcal{B} \underset{\mathrm{R}_X}{\overset{\mathrm{L}_X}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{B}_X^{\mathrm{kl}} \circlearrowleft \mathrm{M}_X$$

with unit

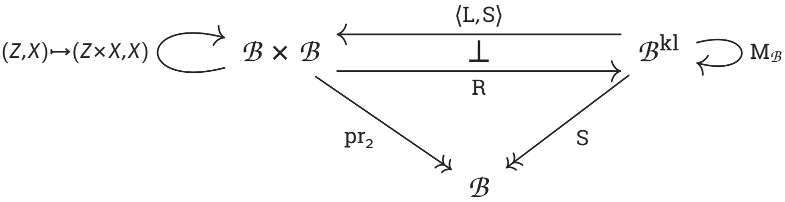
$$Z \xrightarrow{\mathrm{id}_{Z \times X}} Z \times X \qquad \xmapsto{\quad \mathrm{L}_X \quad} \qquad Z \times X \xrightarrow{\mathrm{pr}_{1,2,2}} Z \times X \times X$$

and multiplication

$$Z \times X \times X \xrightarrow{\mathrm{pr}_{1,2}} Z \times X \qquad \xmapsto{\quad \mathrm{L}_X \quad} \qquad Z \times X \times X \times X \xrightarrow{\mathrm{pr}_{1,2,4}} Z \times X \times X$$

The (fibred) reader comonad

$\mathcal{B}$ a category with finite products.

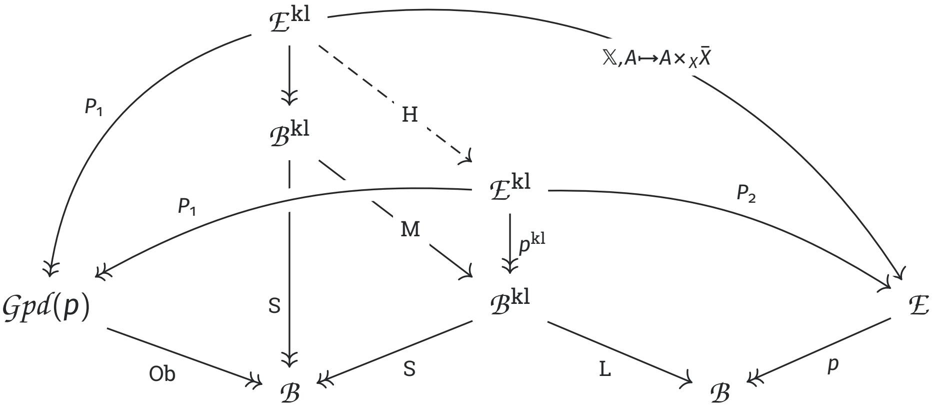


S is a split fibration, called the simple fibration on $\mathcal{B}$.

The fibre of S over X is the Kleisli category of the reader comonad $(-) \times X$.

On a side: $\mathrm{Coalg}((-) \times X) \cong \mathcal{B}/X$, but cod is not a fibration in general.

A monad for p-groupoid actions



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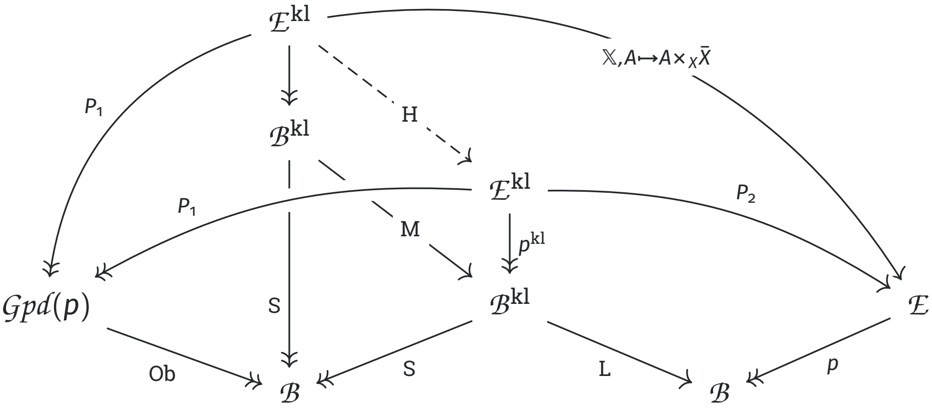

The pair (M, H) is a monad in **Fib** on p^{kl} over Ob: $\mathcal{G}pd(p) \twoheadrightarrow \mathcal{B}$.

A fibration of p-groupoid actions

Obtain a commutative square of fibrations

$$\begin{array}{ccc} \mathcal{Alg}(\mathrm{H}) & \twoheadrightarrow & \mathcal{Gpd}(p) \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow\,\mathrm{Ob} \\ \mathcal{Alg}(\mathrm{M}) & \twoheadrightarrow & \mathcal{B} \end{array}$$

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Algebras of M are triples $(Z, X, v\colon Z \times X \times X \to Z)$ such that

$$z : Z,\ x : X \mid v(z, x, x) = z,$$
$$z : Z,\ x_1 : X,\ x_2 : X,\ x_3 : X \mid v(v(z, x_1, x_2), x_2, x_3) = v(z, x_1, x_3).$$

Actions of a p-groupoid $\mathbb{X}$ are over

$$X \times X \xrightarrow{\mathrm{pr}_2} X \quad = \quad \mathrm{L}\left(1, X, 1 \times X \times X \xrightarrow{!_{X\times X}} 1\right)$$

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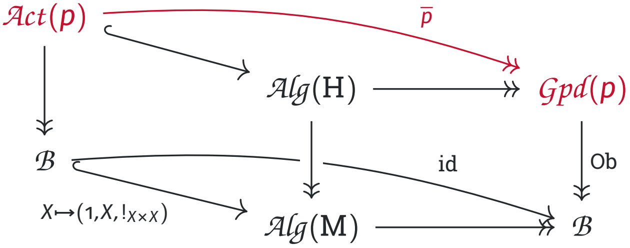

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A fibration of p-groupoid actions - Examples

1. When p is faithful ($\sim$ primary doctrine), $\overline{p}\colon \mathcal{Act}(p) \longrightarrow\!\!\!\!\rightarrow \mathcal{Gpd}(p)$ is ($\sim$) the doctrine of descent data over p-equivalence relations.
2. When $p = \mathcal{Set} \longrightarrow \mathit{1}$, $\mathcal{Gpd}(p) = \mathcal{Grp}$ and $\mathcal{Act}(p)_G = \mathcal{Set}^G$ is the topos of G-sets.
3. When $p = \mathcal{Top} \longrightarrow \mathit{1}$, $\mathcal{Gpd}(p) = \mathcal{TopGrp}$ and $\mathcal{Act}(p)_G$ is the category of continuous G-actions.
4. For $\mathcal{C}$ with finite limits and $p = \mathrm{cod}\colon \mathcal{C}^2 \longrightarrow \mathcal{C}$, $\mathcal{Gpd}(p) = \mathcal{Gpd}(\mathcal{C})$ and objects of $\mathcal{Act}(p)$ are internal actions:[1]

$$\begin{array}{ccccc} A & \longleftarrow & A \times_{G_0} G_1 \lrcorner & \xrightarrow{a} & A \\ \downarrow & & \downarrow & & \downarrow \\ G_0 & \xleftarrow{\mathrm{dom}_G} & G_1 & \xrightarrow{\mathrm{cod}_G} & G_0 \end{array}$$

[1]G. Janelidze, W. Tholen. Facets of Descent II. *Appl. Categ. Struct.* 5, 1997

Monadic p-groupoid actions

$p\colon \mathcal{E} \twoheadrightarrow \mathcal{B}$ an existential elementary doctrine and $\mathbb{X}$ a p-groupoid.

The monad on $\mathrm{H}_{\mathbb{X}}$ induces a monad structure on $\mathcal{E}_X \xrightarrow{\mathrm{H}_{\mathbb{X}}} \mathcal{E}_{X\times X} \xrightarrow{\Sigma_X} \mathcal{E}_X$ and

$$\mathcal{A}lg(\Sigma_X \mathrm{H}_{\mathbb{X}}) \cong \mathcal{A}ct(p)_{\mathbb{X}}$$

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$$\mathcal{Alg}(\Sigma_X \mathrm{H}_{\mathbb{X}}) \cong \mathcal{Act}(p)_{\mathbb{X}}$$

For every $f\colon X \to Y$, there are

- a p-groupoid (X, κ_f) where $\kappa_f = (f \times f)^* \mathrm{I}_Y$ (the kernel p-groupoid), and
- a functor $\sigma_f\colon \mathcal{E}_X \longrightarrow \mathcal{E}_Y$ such that $f^* \sigma_f = \Sigma_X \mathrm{H}_{\kappa_f}$.

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If $\mathcal{B}$ has pullbacks and p has BC for *all* pullbacks, then $\mathcal{A}ct(p)_{\kappa_f} \cong \mathcal{D}es(f)$ and the Bénabou–Roubaud Theorem follows:[2] an arrow f is of effective descent if and only if f^* is monadic.

[2] J. Bénabou, J. Roubaud. Monades et descente. *C. R. Acad. Sc. Paris* 270, 1970

A 2-comonad for elementary fibrations I

The function $p\colon \mathcal{E} \longrightarrow\!\!\!\!\rightarrow \mathcal{B} \mapsto \overline{p}\colon \mathcal{A}ct(p) \longrightarrow\!\!\!\!\rightarrow \mathcal{G}pd(p)$ lifts to a (strict) 2-functor $\overline{(-)}\colon \mathbf{PrdFib} \longrightarrow \mathbf{PrdFib}$.

A morphism of fibrations $p \to p'$ induces

- a lax morphism of monads $(\mathrm{M}_{\mathcal{B}}, \mathrm{H}_p) \to (\mathrm{M}_{\mathcal{B}'}, \mathrm{H}_{p'})$, which induces
- a morphism of fibrations $\mathcal{A}lg(\mathrm{M}_{\mathcal{B}}, \mathrm{H}_p) \to \mathcal{A}lg(\mathrm{M}_{\mathcal{B}'}, \mathrm{H}_{p'})$, which restricts to
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Similarly for 2-cells.

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Theorem (E.–Pasquali–Rosolini)

The 2-functor $\overline{(-)}$ has the structure of a colax-idempotent comonad on $\mathbf{PrdFib}$.

A 2-comonad for elementary fibrations II

Theorem (E.–Pasquali–Rosolini)

The 2-functor $\overline{(-)}$ has the structure of a colax-idempotent comonad on **PrdFib**.

The counit $\varepsilon\colon \overline{(-)} \Longrightarrow \mathrm{Id}_{\mathbf{PrdFib}}$ is given on p by

$$\begin{array}{ccc} \mathcal{A}ct(p) & \longrightarrow & \mathcal{E} \\ {\scriptstyle \overline{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{G}pd(p) & \longrightarrow & \mathcal{B} \end{array} \qquad \begin{array}{ccc} (A, \alpha\colon \mathrm{H}_{\mathbb{X}}A \to A) & \longmapsto & A \\ & & \\ \mathbb{X} & \longmapsto & X \end{array}$$

A coalgebra equips p with the elementary structure.

Theorem (E.–Pasquali–Rosolini)

1. The 2-functor $\overline{(-)}$ lands in **ElFib**.
2. It provides a left 2-adjoint to the forgetful **ElFib** $\longrightarrow$ **PrdFib**.
3. The canonical comparison 2-functor is a 2-equivalence.

A 2-monad for fibrations “with quotients” I

The comonadic 2-adjunction

$$\mathbf{ElFib} \underset{\longrightarrow}{\overset{\overline{(-)}}{\underset{\top}{\longleftarrow}}} \mathbf{PrdFib}$$

induces a lax-idempotent 2-monad on **ElFib** with underlying 2-functor $\overline{(-)}$.

For p elementary, the unit η_p is the coalgebra on p.

A 2-monad for fibrations "with quotients" I

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An algebra is

$$\begin{array}{ccc} \mathcal{A}ct(p) & \xrightarrow{P} & \mathcal{E} \\ {\scriptstyle \overline{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{G}pd(p) & \xrightarrow{Q} & \mathcal{B} \end{array}$$

such that (Q, P) is a reflector for η_p, *i.e.* $(Q, P) \dashv \eta_p$ with invertible counit.
In particular: $Q(X, \mathrm{I}_X) \cong X$, naturally in X.

A 2-monad for fibrations "with quotients" II

For $(f, \bar{f})\colon \mathbb{X} \to (Y, \mathrm{I}_Y)$, the adjunction $Q \dashv ((-), \mathrm{I}_{(-)})$ gives us that the left-hand square commutes iff the right-hand one does:

$$\begin{array}{ccc} Q(X, \mathrm{I}_X) & \xrightarrow{\;\sim\;} & X \\ {\scriptstyle Q(\mathrm{id}, \iota_{\bar{X}})}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Q(\mathbb{X}) & \xrightarrow[(f, \bar{f})^{\#}]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} (X, \mathrm{I}_X) & \xrightarrow{\;\mathrm{id}\;} & (X, \mathrm{I}_X) \\ {\scriptstyle (\mathrm{id}, \iota_{\bar{X}})}\big\downarrow & & \big\downarrow{\scriptstyle (f, \mathrm{I}_f)} \\ \mathbb{X} & \xrightarrow[(f, \bar{f})]{} & (Y, \mathrm{I}_Y) \end{array}$$

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$$\begin{array}{ccc} Q(X, \mathrm{I}_X) & \xrightarrow{\ \sim\ } & X \\ {\scriptstyle Q(\mathrm{id}, \iota_{\bar{X}})}\big\downarrow & \overset{q_{\mathbb{X}}}{\swarrow} & \big\downarrow{\scriptstyle f} \\ Q(\mathbb{X}) & \xrightarrow[(f, \bar{f})^{\#}]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} (X, \mathrm{I}_X) & \xrightarrow{\ \mathrm{id}\ } & (X, \mathrm{I}_X) \\ {\scriptstyle (\mathrm{id}, \iota_{\bar{X}})}\big\downarrow & & \big\downarrow{\scriptstyle (f, \mathrm{I}_f)} \\ \mathbb{X} & \xrightarrow[(f, \bar{f})]{} & (Y, \mathrm{I}_Y) \end{array}$$

i.e. a bijection between morphisms of groupoids $(f, \bar{f})$ as above and arrows $g\colon Q(\mathbb{X}) \to Y$ such that $g \circ q_{\mathbb{X}} = f$.

Moreover:

1. The groupoid $\mathbb{X}$ is the kernel p-groupoid of q, *i.e.* $\mathbb{X} \cong (q \times q)^{*}\mathrm{I}_{Q(\mathbb{X})}$.
2. Every $q_{\mathbb{X}}$ is of effective descent, *i.e.* $\mathcal{E}_{Q(\mathbb{X})} \cong \mathcal{A}ct(p)_{\mathbb{X}}$.

Another 2-comonad

Define a *p-setoid* as a *p*-groupoid but dropping the equations. Then:

- Can construct the endomorphism (M, H) *but* not able to show that it is a monad (the groupoid equations seems to be needed).
- Can still construct a 2-functor $\widehat{(-)}$ on $\mathbf{PrdFib}$ and show that it is a 2-comonad. *But* not able to show that it is colax-idempotent (the groupoid equations seems to be needed).
- What are the coalgebras?
- Similarly, the 2-monad induced on the coalgebras is not lax-idempotent.
- What are the algebras?

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- What are the algebras?

A cofree coalgebra for $\widehat{(-)}$ appears in:
[B. van den Berg, I. Moerdijk. Exact completion of path categories and Algebraic Set Theory. Part I. *J. Pure Appl. Algebra* 222, 2018.]

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