Partial equivalence relations, localizations, and equipments

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Overview

Structure of talk

- The exact completion of a regular hyperdoctrine as a localization
- Derived functors
- Equipments

The exact completion of a regular hyperdoctrine as a localization

Regular hyperdoctrines

Definition

A **regular hyperdoctrine**[1] is a functor

$$\mathcal{H} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Pos} \qquad (\textbf{Pos} \text{ category of posets and monot. maps})$$

such that

- $\mathbb{C}$ has **finite limits**
- all $\mathcal{H}(A)$ have **finite meets**
- for $f : A \to B$, the reindexing map $f^* = \mathcal{H}(f) : \mathcal{H}(B) \to \mathcal{H}(A)$ has a **left adjoint** $\exists_f = f_! : \mathcal{H}(A) \to \mathcal{H}(B)$
- **Frobenius condition**: $(\exists_f \varphi) \wedge \psi \;=\; \exists_f(\varphi \wedge f^* \psi)$ for all $f : A \to B$, $\varphi \in \mathcal{H}(A)$, $\psi \in \mathcal{H}(B)$
- **Beck–Chevalley condition**: $\exists_k h^* \;=\; g^* \exists_f$ for all pullbacks $\begin{array}{ccc} D & \xrightarrow{k} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$

Remarks

- In the article these are called **indexed frames**, but in the spirit of the event, I'm sticking with the 'doctrine' terminology here.
- BC condition stronger than for **elementary existential doctrines** in the sense of[2].

[2] M. Maietti and G. Rosolini. "Unifying exact completions". In: *Applied Categorical Structures* (2012), pp. 1–10

Examples of regular hyperdoctrines

- For X a **locale**, define $\mathcal{H}_X : \mathrm{Set}^{op} \to \mathbf{Pos}$ by

$$\mathcal{H}_X(A) \;=\; (X^A, \leq) \qquad \text{(pointwise ordering)}$$

- Define the **effective tripos** $\mathbf{eff} : \mathrm{Set}^{op} \to \mathbf{Pos}$ by

$$\mathbf{eff}(A) = (P(\mathbb{N})^A, \leq)$$

with $\varphi \leq \psi$ if there exists a *partial recursive* $f : \mathbb{N} \rightharpoonup \mathbb{N}$ such that

$$\forall a \in A \forall n \in \varphi(a) \,.\, f(n) \in \psi(a).$$

- Define the **primitive recursive hyperdoctrine** $\mathbf{prim} : \mathrm{Set}^{op} \to \mathbf{Pos}$ by

$$\mathbf{prim}(A) = (P(\mathbb{N})^A, \leq)$$

with $\varphi \leq \psi$ if there exists a *primitive recursive* $f : \mathbb{N} \to \mathbb{N}$ such that

$$\forall a \in A \forall n \in \varphi(a) \,.\, f(n) \in \psi(a).$$

The exact completion of a regular hyperdoctrine

Definition

The exact completion $\mathbb{C}[\mathcal{H}]$ of a regular hyperdoctrine $\mathcal{H} : \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is given as follows:

- **objects** are pairs $(A \in \mathbb{C}, \rho \in \mathcal{H}(A \times A))$ such that

$$\begin{aligned} (sym)\quad & \rho(x, y) \vdash \rho(y, x) \\ (trans)\quad & \rho(x, y), \rho(y, z) \vdash \rho(x, z) \end{aligned}$$

- **morphisms** $(A, \rho) \to (B, \sigma)$ are predicates $\phi \in \mathcal{H}(A \times B)$ such that

$$\begin{aligned} (strict)\quad & \phi(x, y) \vdash \rho x \wedge \sigma y \qquad [\text{short for } \rho(x, x) \wedge \sigma(y, y)] \\ (cong)\quad & \rho(x, x'), \phi(x', y), \sigma(y, y') \vdash \phi(x, y') \\ (sv)\quad & \phi(x, y), \phi(x, y') \vdash \sigma(y, y') \\ (tot)\quad & \rho x \vdash \exists y \,.\, \phi(x, y) \end{aligned}$$

- **composition** is relational composition — $(\psi \circ \phi)(x, z) \;\equiv\; \exists y \,.\, \phi(x, y) \wedge \psi(y, z)$

Proposition

$\mathbb{C}[\mathcal{H}]$ is a **Barr-exact** category (and a topos, if $\mathcal{H}$ is a tripos).

Moreover, the construction $\mathcal{H} \mapsto \mathbb{C}[\mathcal{H}]$ constitutes a left biadjoint to the forgetful functor from exact categories to regular hyperdoctrines.[3]

[3] M. Maietti and G. Rosolini. “Unifying exact completions”. In: *Applied Categorical Structures* (2012), pp. 1–10

Examples

- $\mathsf{Set}[\mathcal{H}_X] \simeq \mathbf{Sh}(X)$ for any locale X
- $\mathsf{Set}[\mathbf{eff}]$ is the **effective topos**[4] (the best-known *realizablity topos*[5])
- $\mathsf{Set}[\mathbf{prim}]$ is a **list-arithmetic pretopos**[6]

Next we'll give an alternative characterization / universal property of $\mathbb{C}[\mathcal{H}]$, via an intermediate category $\mathbb{C}\langle\mathcal{H}\rangle$.

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[4] J.M.E. Hyland. "The effective topos". In: *The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*. Vol. 110. Stud. Logic Foundations Math. Amsterdam: North-Holland, 1982, pp. 165–216.

[5] J. van Oosten. *Realizability: An Introduction to its Categorical Side*. Elsevier Science Ltd, 2008. ISBN: 0444515844.

[6] M. Maietti. "Joyal's arithmetic universe as list-arithmetic pretopos". In: *Theory and Applications of Categories* 24.3 (2010), pp. 39–83.

The category $\mathbb{C}\langle\mathcal{H}\rangle$

Definition

$\mathbb{C}\langle\mathcal{H}\rangle$ is the category where

- **objects** are pairs $(A \in \mathbb{C}, \rho \in \mathcal{H}(A \times A))$ such that

$$\begin{array}{rl} \textit{(sym)} & \rho(x,y) \vdash \rho(y,x) \\ \textit{(trans)} & \rho(x,y), \rho(y,z) \vdash \rho(x,z) \end{array}$$

- **morphisms** $(A, \rho) \to (B, \sigma)$ are morphisms $f : A \to B$ in $\mathbb{C}$ such that

$$\rho(x,y) \vdash \sigma(fx, fy)$$

- composition and identities are inherited from $\mathbb{C}$

Remark

In[7], $\mathbb{C}\langle\mathcal{H}\rangle$ is characterized as completion of $\mathbb{C}$ under *comprehension*[8] *and* $\mathcal{H}$-descent quotients.

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[8] M. Maietti and G. Rosolini. "Elementary quotient completion". In: *Theory and Applications of Categories* 27 (2012), Paper No. 17, 463
[8] In the talk and the original version of the slides I forgot to mention the comprehension.

The functor from $\mathbb{C}[\mathcal{H}]$ to $\mathbb{C}\langle\mathcal{H}\rangle$

Definition

$$\begin{array}{rcccc} E & : & \mathbb{C}\langle\mathcal{H}\rangle & \to & \mathbb{C}[\mathcal{H}] \\ & & (A,\rho) & \mapsto & (A,\rho) \\ & & \downarrow f & \mapsto & \downarrow \phi \\ & & (B,\sigma) & \mapsto & (B,\sigma) \end{array} \qquad \text{where} \quad \phi(x,y) \equiv \rho x \wedge \sigma(fx,y).$$

Lemma

$f : (A,\rho) \to (B,\sigma)$ is inverted by E iff

$$\text{(inj)} \quad \rho x, \sigma(fx,fy), \rho y \vdash \rho(x,y) \text{ and}$$

$$\text{(esurj)} \quad \sigma u \vdash \exists x\,.\,\rho x \wedge \sigma(fx,u)$$

hold in $\mathcal{H}$.

Definition

Call $f : (A,\rho) \to (B,\sigma)$ a **weak equivalence** if (inj) and (surj) hold.

It turns out that $\mathbb{C}[\mathcal{H}]$ is the **localization** of $\mathbb{C}\langle\mathcal{H}\rangle$ at the weak equivalences.

We-categories

Definition

A **we-category**[9] is a category $\mathcal{C}$ with a class $\mathcal{W} \subseteq \mathrm{mor}(\mathcal{C})$ of **weak equivalences** satisfying **3-for-2**:

(A) For composable arrows $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$, if either two of f, g, and gf are in $\mathcal{W}$, then so is the third.

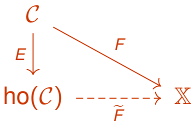
Definition

The **homotopy category** $\mathrm{ho}(\mathcal{C})$ of a we-category $\mathcal{C}$ — also known as **localization** $\mathcal{C}[\mathcal{W}^{-1}]$ of $\mathcal{C}$ at $\mathcal{W}$ — is the category obtained by **freely inverting** the $\mathcal{W}$-arrows in $\mathcal{C}$.

More precisely, precomposition with the **localization functor** $E : \mathcal{C} \to \mathrm{ho}(\mathcal{C})$ induces an isomorphism

$$[\mathrm{ho}(\mathcal{C}), \mathbb{X}] \;\cong\; [\mathcal{C}, \mathbb{X}]_{\mathcal{W}}$$

between the functor category $[\mathrm{ho}(\mathcal{C}), \mathbb{X}]$ and the full subcategory of $[\mathcal{C}, \mathbb{X}]$ on functors sending weak equivalences to isomorphisms.



[9]W.G. Dwyer, P.S. Hirschhorn, D.M. Kan, and J.H. Smith. *Homotopy limit functors on model categories and homotopical categories*. Vol. 113. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2004

The homotopy category of $\mathbb{C}\langle\mathcal{H}\rangle$

Theorem

$E : \mathbb{C}\langle\mathcal{H}\rangle \to \mathbb{C}[\mathcal{H}]$ is a localization, i.e. for every $F : \mathbb{C}\langle\mathcal{H}\rangle \to \mathbb{D}$ inverting weak equivalences, there exists a unique $\tilde{F} : \mathbb{C}[\mathcal{H}] \to \mathbb{D}$ with $\tilde{F} \circ E = F$.

$$\begin{array}{ccc} \mathbb{C}\langle\mathcal{H}\rangle & & \\ {\scriptstyle E}\downarrow & \searrow^{F} & \\ \mathbb{C}[\mathcal{H}] & \dashrightarrow & \mathbb{D} \end{array}$$

Proof (sketch).

$\tilde{F}$ coincides with F on objects. For $[\phi] : (A, \rho) \to (B, \sigma)$ construct the span

$$(A, \rho) \xleftarrow{\phi_l} (A \times B, (\rho \bowtie \sigma)|_{\phi}) \xrightarrow{\phi_r} (B, \sigma)$$

where the underlying maps are projections, and

$$(\rho \bowtie \sigma)|_{\phi}(a, b, a', b') \;\equiv\; \rho(a, a') \wedge \sigma(b, b') \wedge \phi(a, b).$$

Then ϕ_l is a weak equivalence, and $\tilde{F}([\phi])$ is given by

$$\tilde{F}([\phi]) = F(\phi_r) \circ F(\phi_l)^{-1}$$

Derived functors

Motivation

Given a natural transformation $\Phi : \mathcal{H} \to \mathcal{K}$ between regular hyperdoctrines which preserves all of regular logic, the UMP of the localization gives a functor

$$\mathbb{C}[\Phi] : \mathbb{C}[\mathcal{H}] \to \mathbb{C}[\mathcal{K}].$$

However, sometimes we can also construct such a functor if Φ only preserves finite meets.

The following gives an attempt at explaining this using the language of **derived functors**.

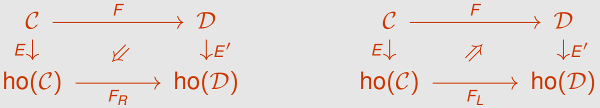
Derived functors

Derived functors are Kan extensions along localizations. Specifically:

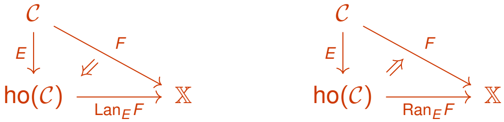
Definition (taken from[10])

Given we-categories $\mathcal{C}$ and $\mathcal{D}$ with localization functors $E : \mathcal{C} \to \text{ho}(\mathcal{C})$, $E' : \mathcal{D} \to \text{ho}(\mathcal{D})$ and a functor $F : \mathcal{C} \to \mathcal{D}$ (not required to preserves we's),

- a **(total) right derived functor** F_R is a **left Kan extension** of $E' \circ F$ along E, and
- a **(total) left derived functor** F_L is a **right Kan extension** of $E' \circ F$ along E.



For **triangular** diagrams, we continue to use the ‘Kan extension’ terminology.



[10] W.G. Dwyer, P.S. Hirschhorn, D.M. Kan, and J.H. Smith. *Homotopy limit functors on model categories and homotopical categories*. Vol. 113. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2004

Kan extensions along localizations

Definition

A functor $f : \mathbb{A} \to \mathbb{B}$ is called **absolutely dense** (a.k.a. **co-fully faithful**), if the precomposition functor

$$(- \circ F) : [\mathbb{B}, \mathbb{X}] \to [\mathbb{A}, \mathbb{X}]$$

is fully faithful for all $\mathbb{X}$.

- From the enriched universal property it it is immediate that localizations $E : \mathcal{C} \to \mathrm{ho}(\mathcal{C})$ are absolutely dense.
- Now given a functor $F : \mathcal{C} \to \mathbb{X}$ such that both Kan extensions exists, we obtain a canonical arrow

  $$\mathrm{Ran}_E F \to \mathrm{Lan}_E F : \mathrm{ho}(\mathcal{C}) \to \mathbb{X}$$

  by dual 'adjoint cylinder style' arguments (analogous to the points-to-pieces transform in cohesion).
- This might explain the left/right switch between Kan extensions and derived functors.

Proto-fibrant objects

Kan extensions along localizations need not always exist. In the following we give a sufficient criterion.

Definition

An object X in a we-category $\mathcal{C}$ is called **proto-fibrant**, if

$$\mathcal{C}(A, X) \to \mathrm{ho}(\mathcal{C})(E(A), E(X))$$

is surjective for all objects A.

$\mathcal{C}$ is said to have **enough proto-fibrant objects**, if every object A admits a weak equivalence $A \xrightarrow{\sim} \overline{A}$ into a proto-fibrant object.

Lemma

Let $\mathcal{C}$ be a we-category with enough proto-fibrant objects, and $F : \mathcal{C} \to \mathbb{D}$ a functor into an arbitrary category, such that

$$Ef = Eg \implies Ff = Fg$$

for all parallel pairs $f, g : A \to X$ with proto-fibrant codomain. Then F admits a left Kan extension along E.

Proto-fibrant and proto-cofibrant objects in $\mathbb{C}\langle\mathcal{H}\rangle$

Definition

Given a regular hyperdoctrine $\mathcal{H} : \mathbb{C}^{op} \to \mathsf{Pos}$, a **tracking map** of $\phi : (A, \rho) \to (B, \sigma)$ in $\mathbb{C}\langle\mathcal{H}\rangle$ is an arrow $f : A \to B$ in $\mathbb{C}$ such that

$$\phi(x, y) \dashv\vdash \rho(x, x) \wedge \sigma(f(x), y).$$

Observation

- (A, ρ) is proto-fibrant in $\mathbb{C}\langle\mathcal{H}\rangle$ if every $\phi : (X, \xi) \to (A, \rho)$ has a tracking map.
- (B, σ) is proto-cofibrant in $\mathbb{C}\langle\mathcal{H}\rangle$ if every $\phi : (B, \sigma) \to (X, \xi)$ has a tracking map.

It turns out that proto-fibrant objects are well known – they are called **weakly complete objects** in[11], and it is shown (using different terminology) that $\mathbb{C}\langle\mathcal{H}\rangle$ has enough proto-fibrant objects whenever $\mathcal{H}$ is a tripos.

[11] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. “Tripos theory”. In: *Math. Proc. Cambridge Philos. Soc.* 88.2 (1980), pp. 205–232.

$\exists$-prime predicates

A sufficient criterion for existence of enough proto-cofibrant objects involves **$\exists$-prime predicates**.

Definition

A predicate $\varpi \in \mathcal{H}(I)$ of a regular hyperdoctrine $\mathcal{H}$ is called **$\exists$-prime**, if for every composable pair $I \xleftarrow{u} J \xleftarrow{v} K$ of maps and every $\psi \in \mathcal{H}(K)$ satisfying $u^*\varpi \leq \exists_v \psi$, there exists a section s of v such that $u^*\varpi \leq s^*\psi$.

Remarks

- The notion of $\exists$-prime predicate is an indexed analogue of the notion of **completely join prime** element of a complete lattice.

  Recall that an element $p \in L$ of a complete lattice is called completely join prime if for any family $(l_i \mid i \in I)$ of elements in L we have

  $$p \leq \bigvee_i l_i \implies \exists i \in I \,.\, p \leq l_i.$$

  Given a complete lattice L, and can show that a predicate in the canonical indexing is $\exists$-prime iff it's pointwise completely join prime.
- $\exists$-primality is called 'existential-freeness' in recent work of Trotta, Maietti, Spadetto, and de Paiva[12],[13].

[13] M. Maietti and D. Trotta. "Generalized existential completions and their regular and exact completions". In: (Nov. 2021). arXiv: 2111.03850 [math.CT]

[13] D. Trotta, M. Spadetto, and V. de Paiva. "Dialectica logical principles". In: *Logical foundations of computer science*. Vol. 13137. Lecture Notes in Comput. Sci. Springer, Cham, [2022] ©2022, pp. 346–363. DOI: 10.1007/978-3-030-93100-1_{2}{2}

$\exists$-prime predicates and proto-cofibrant objects

Definition

Given an regular hyperdoctrine $\mathcal{H} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Pos}$, the **support** of an object $(A, \rho) \in \mathbb{C}\langle\mathcal{H}\rangle$ is the predicate

$$(a{:}A \mid \rho(a, a)) \;=\; \delta_A^* \rho \;\in\; \mathcal{H}(A).$$

Proposition

Given a regular hyperdoctrine $\mathcal{H}$.

- If the support of (A, ρ) is $\exists$-prime then (A, ρ) is proto-cofibrant.
- If $\mathcal{H}$ has **enough $\exists$-prime predicates**[a], then $\mathbb{C}\langle\mathcal{H}\rangle$ has enough proto-cofibrant objects.

[a] in the sense that for every $\phi \in \mathcal{H}(I)$ there exists an $u : J \to I$ and an $\exists$-prime $\varpi \in \mathcal{H}(J)$ with $\exists_u \varpi = \varphi$.

Derived functors

Proposition

Let $\mathcal{H}, \mathcal{K} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Pos}$ be regular hyperdoctrines, and let $\Phi : \mathcal{H} \to \mathcal{K}$ a finite-meet-preserving transformation.

1. If $\mathcal{H}$ has **enough $\exists$-prime predicates** then $\mathbb{C}\langle\Phi\rangle : \mathbb{C}\langle\mathcal{H}\rangle \to \mathbb{C}\langle\mathcal{K}\rangle$ has a **left derived functor**.
2. If $\mathcal{H}$ is a **tripos** then $\mathbb{C}\langle\Phi\rangle : \mathbb{C}\langle\mathcal{H}\rangle \to \mathbb{C}\langle\mathcal{K}\rangle$ has a **right derived** functor.

Examples

1. The right derived functors are used in[a] to construct geometric morphisms between toposes from geometric morphisms between triposes.
2. Let $j : \mathsf{eff} \to \mathsf{eff}$ be the LT topology corresponding to Lifschitz realizability. Then the right derived functor is the reflection onto the subtopos, and the left derived functor is the reflection onto separated objects.

[a] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. "Tripos theory". In: *Math. Proc. Cambridge Philos. Soc.* 88.2 (1980), pp. 205–232.

Equipments

Right derived functors are oplax functorial

Definition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between we-categories with enough proto-fibrant objects.

- Call F **congruent**, if
$$Ef = Eg \implies E'(Ff) = E'(Fg)$$
for all parallel arrows $f, g : A \to X$ with proto-fibrant codomain.
- Call F **exact**, if it preserves weak equivalences.

Proposition

- Given congruent functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{D}$ between we-categories with enough proto-fibrant objects, there's a natural transformation
$$\xi_{F,G} : (G \circ F)_R \to G_R \circ F_R$$
induced by the UMP of $(G \circ F)_R$.
- $\xi_{F,G}$ is invertible whenever G is exact.

This looks like a morphism of equipments!

Definition

1. A **equipment** is a 2-category $\mathfrak{C}$ equipped with a subcategory $\mathfrak{C}_r$ of **regular** 1-cells.
2. An **special functor** between equipments is an oplax functor $F : \mathfrak{C} \to \mathfrak{D}$ such that Ff is regular whenever f is regular, all $F\mathrm{id}_A \to \mathrm{id}_{FA}$ are invertible, and $F(gf) \to Fg\, Ff$ is invertible whenever g is regular.
3. A **special transformation** is an oplax transformation $\eta : F \to G$ such that all η_A are regular and $\eta_B\, Ff \to Gf\, \eta_A$ is invertible whenever f is regular.

Equipments form a 2**-category enriched category** in the sense of Verity[14], and we have

Proposition

The assignments $\mathcal{C} \mapsto \mathrm{ho}(\mathcal{C})$ and $F \mapsto F_R$ give rise to a **special left biadjoint** to the inclusion

$$\mathbf{Cat} \hookrightarrow \mathbf{PF\text{-}weCat}$$

of **Cat** into the equipment of we-categories with enough proto-fibrant objects (with exact functors as 1-cells and congruent functors as special 1-cells).

This can be seen as a conceptual explanation of the result from[15] that the tripos-to-topos construction is a special left biadjoint to the inclusion of triposes into toposes.

[14] D. Verity. "Enriched categories, internal categories and change of base". In: *Repr. Theory Appl. Categ.* 20 (2011), pp. 1–266.
[15] J. Frey. "Triposes, q-toposes and toposes". In: *Annals of Pure and Applied Logic* 166.2 (2015), pp. 232–259.

Bonus: Yet another characterization of $\mathbb{C}[\mathcal{H}]$

- We have shown that the **exact completion** of $\mathcal{H}$ can be described as a localization of the completion of $\mathbb{C}$ under comprehension and $\mathcal{H}$**-descent quotients.**
- Alternatively, the **regular completion** of $\mathcal{H}$ can be characterized as localization of the **total category** $\int \mathcal{H}$ at the arrows $f : (A, \phi) \to (B, \psi)$ such that

$$\phi(a) \vdash \psi(ba)$$
$$\phi(a), \phi(a'), fa = fa' \vdash a = a'$$
$$\psi(b) \vdash \exists a \,.\, \phi(a) \wedge fa = b$$

- thus, the exact completion construction of a regular hyperdoctrine can be decomposed either as

add comprehension $\to$ add descent quotients $\to$ localize

or as

add comprehension $\to$ localize $\to$ ex/reg completion

This is due to Mathieu Anel.

Related work

Related work

- J. van Oosten. “A notion of homotopy for the effective topos”. In: *Math. Structures Comput. Sci.* 25.5 (2015), pp. 1132–1146
- B. van den Berg. “Univalent polymorphism”. In: *Annals of Pure and Applied Logic* 171.6 (2020), pp. 102793, 29. ISSN: 0168-0072. DOI: 10.1016/j.apal.2020.102793

Thanks to Benno for helpful discussions!

Thanks for your attention!