Bidirectional models of Radically Synthetic Differential Geometry

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1. Euler reals

2. Models

"Radically synthetic" smooth geometry

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Therefore we postulate a pointed space T and call the map space X^T the tangent bundle of any space X, with evaluation at the point inducing the bundle map $X^T \to X$.”

The subspace of Euler reals

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Thus R has the intrinsic structure of a monoid with 0. Moreover, it has always been commutative."

Connected components (cont.)

To continue with the axiomatic development a subcategory with a reflector $X \to \pi_0(X)$ satisfying

$$\pi_0(X \times Y) \cong \pi_0(X) \times \pi_0(Y)$$

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Proposition 1

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Proposition 1

$\pi_0(X^T) = \pi_0(X)$ for all X iff $\pi_0(R) = 1$.

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However the intuition for the the real case involves a line which is bi-directional, so that

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In all cases, we can consider U_+ defined as the kernel of the natural homomorphism $U \to \pi_0(U)$ (i.e. the component of the identity) as the group of positive elements of R."

The subgroup of positive units

$$\begin{array}{ccc} U_+ & \longrightarrow & \mathbf{1} \\ \big\downarrow & & \big\downarrow{\scriptstyle 1} \\ U & \longrightarrow & \pi_0 U \end{array}$$

(Lawvere discusses axioms implying that R has an intrinsic addition but let us assume that R has an addition.)

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Define

$$A = \{a \in R \mid a + U_+ \subseteq U_+\}$$

$$R_+ = \{\lambda \in R \mid \lambda A \subseteq A\}$$

Proposition 2 (more general in the paper)

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A is an additive monoid and hence R_+ is a subrig of R.

The (pre-)order

The relation defined by

$$r \leq s \text{ iff } \exists(m \in R_+)[r + m = s]$$

has the expected properties of an ordering.

"The elements h for which

$$0 \leq h \quad \& \quad h \leq 0$$

constitute an ideal that contains all nilpotent quantities."

Summary, starting from $0 : \mathbf{1} \to T$

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U_+ & \longrightarrow & R_+ \\
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(*) Needs $(R, +)$.

Models

Models of SDG

1. (Classical Algebraic Geometry) $\pi_0 U$ connected.
2. ('well-adapted' models) Not clear what is π_0.

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"To clarify the above considerations, generalize to [extensive] categories and seek philosophical guidance."

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If $\mathcal{A}$ is extensive with 1, let $\mathcal{C} \to \mathcal{A}$ be the full subcategory of connected objects that have a point.

Corollary ($\widehat{\mathcal{C}}$ has π_0)

$\widehat{\mathcal{C}} \to$ **Set** *is pre-cohesive. In particular, the canonical* **Set** $\to \widehat{\mathcal{C}}$ *has a left adjoint that preserves finite products.*

A pre-cohesive topos with a bidirectional ring of Euler reals

Let $\mathcal{A}$ be the opposite of the category of finitely generated C^∞-rings (affine C^∞ schemes).

$\mathcal{A}$ is extensive and has terminal object.

Let $\mathcal{C} \to \mathcal{A}$ be the subcategory determined by the connected objects with some point. (So that $\widehat{\mathcal{C}}$ has π_0.)

Let T in $\widehat{\mathcal{C}}$ be the object determined by $\mathbb{R}[x]/(x^2) = \mathbb{R}[\epsilon]$.
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(T has a unique point.)

Proposition

$\widehat{\mathcal{C}}$ embeds the the category of connected manifolds.
The ring of Euler reals determined by T in $\widehat{\mathcal{C}}$ coincides with $\mathbb{R}$ and it is bidirectional.

What is $M \to R$?

A second model

Manifolds with boundary

From [Kock's SDG, III.9]:

Manifolds with boundary

From [Kock's SDG, III.9]: "To get toposes in which the category of manifolds with boundary is nicely and fully embedded, it seems necessary to construct 'smaller' well-adapted models, by choosing suitable full subcategories [of the standard site] as our site of definition.

coW-determined objects

Let $\mathcal{A}$ be an extensive category with 1.
We can consider the objects X such that

1. X has a point or
2. (coW) X has exactly one point or
3. ('coW-determined') The family of maps with codomain X and co-W domain is jointly epic.

Let $\mathcal{D} \to \mathcal{C} \to \mathcal{A}$ be the subcategory of (connected, with some point, and) co-W-determined objects.

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In particular: Let $\mathcal{A}$ be the opposite of the category of finitely generated C^∞-rings (affine C^∞-schemes).

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'Proposition'

The assignment $K \mapsto C^\infty(K)$ is 'good' when considered from connected manifolds-with-boundary to $\mathcal{D}$.

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Theorem

The ring R of Euler reals determined by T in $\widehat{\mathcal{D}}$ coincides with $\mathbb{R}$ and it is bidirectional.
Also, $M \to R$ coincides with $[0, \infty) \to R$.

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