Implicative algebras: first-order completeness

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Different notions of models

- **Tarski models:** $[\![\varphi]\!] \in \{0;1\}$
  - Interprets classical provability (correctness/completeness)
- **Cohen forcing:** $[\![\varphi]\!] \in \mathfrak{P}(P)$ [Cohen '63]
  - Negation of continuum hypothesis, Solovay's axiom, etc.
- **Boolean-valued models:** $[\![\varphi]\!] \in \mathcal{B}$ [Scott, Solovay, Vopěnka]

\+ **Heyting-valued models:** $[\![\varphi]\!] \in \mathcal{H}$ (intuitionistic forcing, Kripke forcing)

- **Intuitionistic realizability:** $[\![\varphi]\!] \in \mathfrak{P}(\Lambda)$ [Kleene '45]
  - Truth value = set of realizers (programs)
  - Incompatible with classical logic
- **Classical realizability:** $[\![\varphi]\!] \in \mathfrak{P}(\Pi)$ [Krivine '94, '01, '09, '11]
  - Interprets classical proofs
  - Generalizes Tarski models and Cohen forcing

Which algebra of truth values

What is an algebra of truth values? (for intuitionistic and classical logic)

- A complete Heyting algebra? (Kripke forcing)
- A complete Boolean algebra? (Cohen forcing)
- A partial combinatory algebra? (intuitionistic realizability)
- An abstract Krivine structure? (classical realizability)

A unifying structure:

Implicative algebra	=	implicative structure	+	separator
		(algebra of truth values)		(criterion of truth)

- Streicher'13: *Krivine's classical realizability from a categorical perspective*
- Ferrer *et al* '17: *Ordered combinatory algebras and realizability*
- M. '20: *Implicative algebras: a new foundation for realizability and forcing*

Implicative algebras

Definition (Implicative structures & algebras) [M. 2020]

1. An implicative structure is a complete lattice $(\mathscr{A}, \preccurlyeq)$ equipped with a binary operation $(\to) : \mathscr{A}^2 \to \mathscr{A}$ such that:

   (1) If $a' \preccurlyeq a$ and $b \preccurlyeq b'$, then $(a \to b) \preccurlyeq (a' \to b')$

   (2) For all $a \in \mathscr{A}$ and $B \subseteq \mathscr{A}$, we have: $a \to \bigcurlywedge_{b \in B} b = \bigcurlywedge_{b \in B}(a \to b)$

2. A separator of $(\mathscr{A}, \preccurlyeq, \to)$ is a subset $S \subseteq \mathscr{A}$ such that:

   (1) If $a \in S$ and $a \preccurlyeq a'$, then $a' \in S$

   (2) $\bigcurlywedge_{a,b \in \mathscr{A}}(a \to b \to a) \;\; (= \mathbf{K}^{\mathscr{A}}) \in S$ and
   $\bigcurlywedge_{a,b,c \in \mathscr{A}}((a \to b \to c) \to (a \to b) \to a \to c) \;\; (= \mathbf{S}^{\mathscr{A}}) \in S$

   (3) If $(a \to b) \in S$ and $a \in S$, then $b \in S$

3. An implicative algebra is an implicative structure $(\mathscr{A}, \preccurlyeq, \to)$ together with a separator $S \subseteq \mathscr{A}$

Examples of implicative algebras

- **Complete Heyting algebras** $(\mathscr{A}, \preccurlyeq)$, letting:

$$\begin{aligned} a \to b &:= \max\{c \in \mathscr{A} \mid (c \curlywedge a) \preccurlyeq b\} \\ S &:= \{\top\} \end{aligned}$$

(Heyting's implication)

+ **Complete Boolean algebras** (as a particular case of cHAs)

- Given a (total) combinatory algebra $(P, \cdot, \mathtt{k}, \mathtt{s})$, we let:
  - $\mathscr{A} := \mathfrak{P}(P)$
  - $a \preccurlyeq b := a \subseteq b$
  - $a \to b := \{x \in P \mid \forall z \in a,\ x \cdot z \in b\}$ (Kleene's implication)
  - $S := \mathscr{A} \setminus \{\varnothing\}$

- Given an abstract Krivine structure $(\mathbf{\Lambda}, \mathbf{\Pi}, \mathbf{\Lambda} \star \mathbf{\Pi}, \ldots, \perp\!\!\!\perp)$, we let:
  - $\mathscr{A} := \mathfrak{P}(\mathbf{\Pi})$
  - $a \preccurlyeq b := a \supseteq b$
  - $a \to b := a^{\perp\!\!\!\perp} \cdot b$ (Krivine's implication)
  - $S := \{a \in \mathfrak{P}(\mathbf{\Pi}) \mid a^{\perp\!\!\!\perp} \cap \mathsf{PL} \neq \varnothing\}$

Defining other logical constructions

- Each implicative algebra $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to, S)$ is basically a model of minimal 1st-order logic, in which

$$(\Rightarrow) = (\to) \qquad \text{and} \qquad (\forall) = (\curlywedge) \qquad \text{(infinitary meets)}$$

- In this setting, other connectives and quantifiers are recovered using standard 2nd-order encodings:

(Negation) $\neg a \;:=\; a \to \bot \qquad (\bot := \min \mathscr{A})$

(Conjunction) $a \times b \;:=\; \displaystyle\bigcurlywedge_{c \in \mathscr{A}} ((a \to b \to c) \to c)$

(Disjunction) $a + b \;:=\; \displaystyle\bigcurlywedge_{c \in \mathscr{A}} ((a \to c) \to (b \to c) \to c)$

(Universal quant.) $\displaystyle\bigforall_{i \in I} a_i \;:=\; \bigcurlywedge_{i \in I} a_i$

(Existencial quant.) $\displaystyle\bigexists_{i \in I} a_i \;:=\; \bigcurlywedge_{c \in \mathscr{A}} \Big(\Big(\bigcurlywedge_{i \in I} (a_i \to c) \Big) \to c \Big)$

Viewing truth values as generalized realizers: a manifesto

1. Elements of an implicative structure are primarily intended to represent truth values. But since λ-abstraction and application are definable in such a structure (cf next slide), we can see:
   - each realizer as a particular truth value;
   - each truth value as a generalized realizer

2. So that we get the ultimate Curry-Howard identification:

   Realizer $=$ **Program** $=$ **Formula** $=$ **Type**

3. In this setting, the relation $a \preccurlyeq b$ may read:
   - a is a subtype of b (viewing a and b as truth values)
   - a has type b (viewing a as a realizer, b as a truth value)
   - a is more defined than b (viewing a and b as realizers)

4. In particular: subtyping ($\preccurlyeq$) $=$ reverse Scott ordering ($\sqsupseteq$)

Representing λ-terms as truth values

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ be an implicative structure

1. Given $a, b \in \mathscr{A}$ and a function $f : \mathscr{A} \to \mathscr{A}$, we let:

$$ab \;:=\; \bigwedge\{c \in \mathscr{A} \mid a \preccurlyeq (b \to c)\} \qquad \text{(application)}$$

$$\boldsymbol{\lambda} f \;:=\; \bigwedge_{a \in \mathscr{A}} (a \to f(a)) \qquad \text{(abstraction)}$$

Remark: Both constructions are monotonic w.r.t. a, b and f

2. Fundamental adjunction: $ab \preccurlyeq c \quad \Leftrightarrow \quad a \preccurlyeq (b \to c)$

3. Therefore, each λ-term t (with parameters in $\mathscr{A}$) can be represented by a truth value $t^{\mathscr{A}} \in \mathscr{A}$ (in a non-necessarily injective way):
   - β-rule: If $t \twoheadrightarrow_\beta t'$, then $(t)^{\mathscr{A}} \preccurlyeq (t')^{\mathscr{A}}$
   - η-rule: If $t \twoheadrightarrow_\eta t'$, then $(t)^{\mathscr{A}} \succcurlyeq (t')^{\mathscr{A}}$

4. We have $t^{\mathscr{A}} \in S$ for all separators S (i.e. all proofs are "true")

Particular case: $\mathscr{A}$ is a complete Heyting algebra

Complete Heyting/Boolean algebras are the particular implicative structures $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ where $\to$ is defined from $\preccurlyeq$ by

$$a \to b \ := \ \max\{c \in \mathscr{A} \mid (c \curlywedge a) \preccurlyeq b\}$$

Remark: Complete Heyting/Boolean algebras are the structures underlying forcing (in the sense of Kripke or Cohen)

Proposition

When $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ is a complete Heyting/Boolean algebra:

1. For all $a, b \in \mathscr{A}$: $ab = a \curlywedge b$ (application $=$ binary meet)
2. For each closed λ-term t: $(t)^{\mathscr{A}} = \top$
3. In this setting, a separator is the same as a filter

But in the general case, separators are *not* filters (since not closed under $\curlywedge$)

Constructions

- Given an implicative algebra $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to, S)$, the separator $S \subseteq \mathscr{A}$ induces a preorder of entailment:

$$a \vdash_S b \quad :\equiv \quad (a \to b) \in S \qquad\qquad \text{(for all } a, b \in \mathscr{A}\text{)}$$

  The poset reflection of $(\mathscr{A}, \vdash_S)$, written $\mathscr{A}/S$, is a Heyting algebra

  (Intuition: $\mathscr{A}/S$ represents the "logic of $\mathscr{A}$")

- More generally, implicative structures (algebras) can be manipulated almost the same way as complete Heyting/Boolean algebras:
  - The product of a family of implicative structures (resp. algebras) is an implicative structure (resp. algebra)
  - The analogue of utrafilters are ultraseparators

$$U \subseteq \mathscr{A} \text{ ultraseparator} \quad :\equiv \quad U \text{ maximal proper separator}$$
$$\Leftrightarrow \quad \mathscr{A}/U \approx \mathbf{2} \quad (= \{0, 1\})$$

    Each proper separator $S \subsetneq \mathscr{A}$ extends to an ultraseparator $U \supseteq S$

The implicative tripos

- Each implicative algebra $(\mathscr{A}, \preccurlyeq, \to, S_{\mathscr{A}})$ induces a ($\mathbf{Set}$-based) tripos $\mathbf{P}_{\mathscr{A}} \ : \ \mathbf{Set}^{\mathsf{op}} \to \mathbf{HA}$:

$$\mathbf{P}_{\mathscr{A}}(I) \ := \ \mathscr{A}^I / S[I] \qquad (I \in \mathbf{Set})$$

where $\quad S[I] \ := \ \{(a_i)_{i \in I} \in \mathscr{A}^I \mid \exists s \in S,\ \forall i \in I,\ s \preccurlyeq a_i\} \ \subseteq \ S^I$

Uniform I-indexed power separator

- This construction encompasses all $\mathbf{Set}$-based triposes known so far:
  - Forcing triposes, which correspond to the case where $(\mathscr{A}, \preccurlyeq, \to)$ is a complete Heyting/Boolean algebra, and $S = \{\top\}$ (i.e. no quotient)
  - Triposes induced by total combinatory algebras... (int. realizability)
    ... and even by partial combinatory algebras, via some completion trick
  - Triposes induced by abstract Krivine structures (class. realizability)

Higher-order completeness

Do implicative triposes encompass all $\mathbf{Set}$-based triposes? Yes!

Theorem (Higher-order completeness) [M. 2020]

Each $\mathbf{Set}$-based tripos is (isomorphic to) an implicative tripos

- Implicative algebras are thus a way to represent $\mathbf{Set}$-based triposes
- From the above result, we deduce that all $\mathbf{Set}$-based triposes are also generated by O(P)CAs with "filters"

(Indeed, these structures have the same expressiveness as implicative algebras)

Moreover, since classical implicative algebras have the same expressiveness as abstract Krivine structures, we deduce that:

Corollary

Each classical $\mathbf{Set}$-based tripos is (isomorphic to) a Krivine tripos

Implicative models of 1st-order theories (1/4)

Given an implicative structure (or algebra) $\mathscr{A}$:

Definition ($\mathscr{A}$-valued model of a 1st-order language $\mathscr{L}$)

An $\mathscr{A}$-valued model $\mathscr{M}$ of a 1st-order language $\mathscr{L}$ is given by:

- a nonempty set M (the domain of interpretation)
- a function $f^{\mathscr{M}} : M^k \to M$ for each k-ary function symbol f
- a function $p^{\mathscr{M}} : M^k \to \mathscr{A}$ for each k-ary predicate symbol p

An implicative model of the language $\mathscr{L}$ is an $\mathscr{A}$-valued model of $\mathscr{L}$ for some implicative structure (or algebra) $\mathscr{A}$

Implicative models of 1st-order theories (2/4)

Definition (Interpretation of terms and formulas with parameters in $\mathscr{L}$)

Given an $\mathscr{A}$-valued model $\mathscr{M}$ of a 1st-order language $\mathscr{L}$:

1. Each closed term t with parameters in $\mathscr{M}$ is interpreted by an element $t^{\mathscr{M}} \in M$, that is defined the usual way
2. Each closed formula φ with parameters in $\mathscr{M}$ is interpreted by a truth value $[\![\varphi]\!]^{\mathscr{M}} \in \mathscr{A}$ that is defined by:

$$[\![p(t_1,\dots,t_k)]\!]^{\mathscr{M}} := p^{\mathscr{M}}(t_1^{\mathscr{M}},\dots,t_k^{\mathscr{M}})$$

$$\begin{array}{rclrcl} [\![\bot]\!]^{\mathscr{M}} & := & \bot_{\mathscr{A}} & [\![\varphi \Rightarrow \psi]\!]^{\mathscr{M}} & := & [\![\varphi]\!]^{\mathscr{M}} \to [\![\psi]\!]^{\mathscr{M}} \\ [\![\varphi \wedge \psi]\!]^{\mathscr{M}} & := & [\![\varphi]\!]^{\mathscr{M}} \times [\![\psi]\!]^{\mathscr{M}} & [\![\varphi \vee \psi]\!]^{\mathscr{M}} & := & [\![\varphi]\!]^{\mathscr{M}} + [\![\psi]\!]^{\mathscr{M}} \\ [\![\forall x\, \varphi]\!]^{\mathscr{M}} & := & \displaystyle\bigwedge_{a\in M} [\![\varphi[x:=a]]\!]^{\mathscr{M}} & [\![\exists x\, \varphi]\!]^{\mathscr{M}} & := & \displaystyle\exists_{a\in M} [\![\varphi[x:=a]]\!]^{\mathscr{M}} \end{array}$$

3. When $\mathscr{A}$ is an implicative algebra (i.e. given with a separator $S \subseteq \mathscr{A}$), we write $\mathscr{M} \models \varphi$ when $[\![\varphi]\!]^{\mathscr{M}} \in S$

Implicative models of 1st-order theories (3/4)

In what follows, we restrict to classical 1st-order theories and classical implicative models (i.e. based on classical implicative algebras)

Definition (Implicative model of a 1st-order theory $\mathscr{T}$)

An implicative model of a 1st-order theory $\mathscr{T}$ is an $\mathscr{A}$-model $\mathscr{M}$ of the language of $\mathscr{T}$ (for some classical implicative algebra $\mathscr{A}$) such that

$$\mathscr{M} \models \varphi \qquad \text{for each axiom of } \mathscr{T}$$

(i.e. $[\![\varphi]\!]^{\mathscr{M}} \in S_{\mathscr{A}}$)

Writing $S_{\mathscr{A}}$ the separator of $\mathscr{A}$

Remark: Boolean-valued (and Tarski) models of $\mathscr{T}$ are particular cases of implicative models of $\mathscr{T}$, where $\mathscr{A}$ is a complete BA $\quad (+\ S_{\mathscr{A}} := \{\top\})$

Proposition (Soundness)

If $\mathscr{T} \vdash \varphi$, then $\mathscr{M} \models \varphi$ in all implicative models $\mathscr{M}$ of $\mathscr{T}$

Implicative models of 1st-order theories (4/4)

Remarks:

- In the definition of the notion of $\mathscr{A}$-model $\mathscr{M}$, the separator $S_{\mathscr{A}} \subseteq \mathscr{A}$ is only used to define the notion of satisfaction:

$$\mathscr{M} \models \varphi \qquad \text{iff} \qquad [\![\varphi]\!]^{\mathscr{M}} \in S_{\mathscr{A}}$$

  (Interpretation $[\![\varphi]\!]^{\mathscr{M}}$ only depends on underlying implicative structure)

- Given an implicative algebra $\mathscr{A}$ and a separator $S' \supseteq S_{\mathscr{A}}$, write $\mathscr{A}{:}S'$ the implicative algebra obtained by replacing $S_{\mathscr{A}}$ by S'

- Similarly, given an $\mathscr{A}$-model $\mathscr{M}$ of $\mathscr{L}$ and a separator $S' \supseteq S_{\mathscr{A}}$, write $\mathscr{M}{:}S'$ the $\mathscr{A}{:}S'$-model obtained by replacing $S_{\mathscr{A}}$ by S'

- Clearly: $\mathscr{M} \models \varphi$ (i.e. $[\![\varphi]\!]^{\mathscr{M}} \in S_{\mathscr{A}}$) implies $\mathscr{M}{:}S' \models \varphi$ (i.e. $[\![\varphi]\!]^{\mathscr{M}} \in S'$)

Full implicative models

Definition (Full implicative models)

An $\mathscr{A}$-valued model $\mathscr{M}$ of a 1st-order language $\mathscr{L}$ is full when for all formulas $\varphi(x_0, x_1, \ldots, x_n) \in \mathscr{L}$ and for all parameters $u_1, \ldots, u_n \in M$, there exists a parameter $u_0 \in M$ such that

$$\mathscr{M} \models \varphi(u_0, u_1, \ldots, u_n) \Rightarrow \forall x\, \varphi(x, u_1, \ldots, u_n)$$

that is: $$\Big([\![\varphi(u_0, u_1, \ldots, u_n)]\!]^{\mathscr{M}} \to \bigcurlywedge_{u \in M} [\![\varphi(u, u_1, \ldots, u_n)]\!]^{\mathscr{M}}\Big) \in S_{\mathscr{A}}$$

Remarks:

- When $\mathscr{A}$ is a complete Boolean algebra, this criterion is equivalent to the usual criterion of fullness for Boolean-valued models
- Recall that all Tarski models are full (by construction)

From classical implicative models to Tarski models

- Given an implicative structure $\mathscr{A}$, recall that an ultraseparator of $\mathscr{A}$ is any maximal consistent separator $U \subseteq \mathscr{A}$. Equivalently:

$$\text{A separator } S \subseteq \mathscr{A} \text{ is an ultraseparator} \quad \text{iff} \quad \mathscr{A}/S \simeq \mathbf{2}$$

- In any implicative structure $\mathscr{A}$, any consistent separator $S \subsetneq \mathscr{A}$ can be extended into an ultraseparator $S \subseteq U \subsetneq \mathscr{A}$

Proposition (Turning implicative models into Tarski models)

Let $\mathscr{M}$ be a full and consistent implicative model of a 1st-order theory $\mathscr{T}$

Then for each ultraseparator $U \supseteq S_{\mathscr{A}}$, $\mathscr{M}{:}U$ is a Tarski model of $\mathscr{T}$

Remark: When $\mathscr{T}$ has an equality symbol, we also need to quotient the model $\mathscr{M}{:}U$ by the equivalence relation $\sim$ induced by equality in $\mathscr{M}{:}U$

Strong completeness for implicative models

Theorem (Strong completeness for implicative models)

(1) For each 1st-order language $\mathscr{L}$, there is an implicative structure $\mathscr{A}$ together with an $\mathscr{A}$-model $\mathscr{M}$ of $\mathscr{L}$ such that for all $\varphi \in \mathscr{L}$:

$$\vdash_{\mathsf{LK}} \varphi \qquad \text{iff} \qquad [\![\varphi]\!]^{\mathscr{M}} \in S_K^0(\mathscr{A})$$

where $S_K^0(\mathscr{A})$ is the classical core (= smallest classical separator) of $\mathscr{A}$

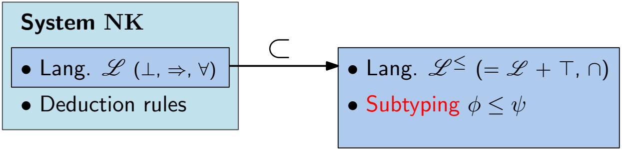
(2) For each 1st-order theory $\mathscr{T}$ on the language $\mathscr{L}$, there is a classical separator $S \subseteq \mathscr{A}$ such that for all $\varphi \in \mathscr{L}$:

$$\mathscr{T} \vdash \varphi \qquad \text{iff} \qquad [\![\varphi]\!]^{\mathscr{M}} \in S$$

(3) We can always choose $\mathscr{A}$ and S in such a way that $\mathscr{M}$ is a full $\mathscr{A}{:}S$-model of the theory $\mathscr{T}$

- Actually, the strong completeness theorem already holds for Boolean-valued models (= particular cases of implicative models)
- Here we give a more "proof-theoretic" / "realizability" argument

Sketch of the proof of strong completeness (1/11)



$$\phi, \psi \in \mathscr{L}^{\leq} \quad ::= \quad \bot \quad | \quad \top \quad | \quad p(t_1, \ldots, t_k) \quad | \quad \phi \cap \psi \quad | \quad \phi \Rightarrow \psi \quad | \quad \forall x\, \phi$$

Sketch of the proof of strong completeness (2/11)

$$\frac{}{\varphi \leq \varphi}\ (\textsc{Refl}) \qquad \frac{\varphi \leq \psi \quad \psi \leq \chi}{\varphi \leq \chi}\ (\textsc{Trans})$$

$$\frac{\varphi' \leq \varphi}{\varphi \Rightarrow \psi \leq \varphi' \Rightarrow \psi}\ (\Rightarrow_{\mathrm{L}}) \qquad \frac{\chi \leq \varphi \Rightarrow \psi \quad \psi \leq \psi'}{\chi \leq \varphi \Rightarrow \psi'}\ (\Rightarrow_{\mathrm{R}})$$

$$\frac{}{\bot \leq \chi}\ (\bot_{\mathrm{L}}) \qquad \frac{}{\chi \leq \top}\ (\top_{\mathrm{R}}) \qquad \frac{}{\chi \leq \varphi \Rightarrow \top}\ (\Rightarrow\top_{\mathrm{R}})$$

$$\frac{\varphi_1 \leq \chi}{\varphi_1 \cap \varphi_2 \leq \chi}\ (\cap^1_{\mathrm{L}}) \qquad \frac{\varphi_2 \leq \chi}{\varphi_1 \cap \varphi_2 \leq \chi}\ (\cap^2_{\mathrm{L}})$$

$$\frac{\chi \leq \varphi_1 \quad \chi \leq \varphi_2}{\chi \leq \varphi_1 \cap \varphi_2}\ (\cap_{\mathrm{R}}) \qquad \frac{\chi \leq \psi \Rightarrow \varphi_1 \quad \chi \leq \psi \Rightarrow \varphi_2}{\chi \leq \psi \Rightarrow \varphi_1 \cap \varphi_2}\ (\Rightarrow\cap_{\mathrm{R}})$$

$$\frac{\varphi[x := t] \leq \chi}{\forall x\, \varphi \leq \chi}\ (\forall_{\mathrm{L}}) \qquad \frac{\chi \leq \varphi}{\chi \leq \forall x\, \varphi}\ (\forall_{\mathrm{R}})\ \text{provided } x \notin FV(\chi)$$

$$\frac{\chi \leq \psi \Rightarrow \varphi}{\chi \leq \psi \Rightarrow \forall x\, \varphi}\ (\Rightarrow\forall_{\mathrm{R}})\ \text{provided } x \notin FV(\chi, \psi)$$

Theorem (Cut elimination)

The rule $(\textsc{Trans})$ is admissible (in the remaining set of rules)

Sketch of the proof of strong completeness (3/11)

System NK

- Lang. $\mathscr{L}$ $(\bot, \Rightarrow, \forall)$
- Deduction rules

$\subset$ →

System $\lambda\mathbf{NK}_0^{\leq}$

- Lang. $\mathscr{L}^{\leq}$ $(= \mathscr{L} + \top, \cap)$
- Subtyping $\phi \leq \psi$ (+ *cut elimination*)
- Proof-terms $M : \phi$

$$\phi, \psi \in \mathscr{L}^{\leq} \quad ::= \quad \bot \quad | \quad \top \quad | \quad p(t_1, \ldots, t_k) \quad | \quad \phi \cap \psi \quad | \quad \phi \Rightarrow \psi \quad | \quad \forall x\, \phi$$

$$M, N \quad ::= \quad \xi \quad | \quad \lambda\xi\,.\,M \quad | \quad M\,N \quad | \quad \mathcal{C}\xi\,.\,M$$

Sketch of the proof of strong completeness (4/11)

Typing rules for proof terms

$$\frac{}{\Gamma \vdash \xi : \varphi}\ (\textsc{Var}) \quad \text{(if } (\xi : \varphi) \in \Gamma\text{)}$$

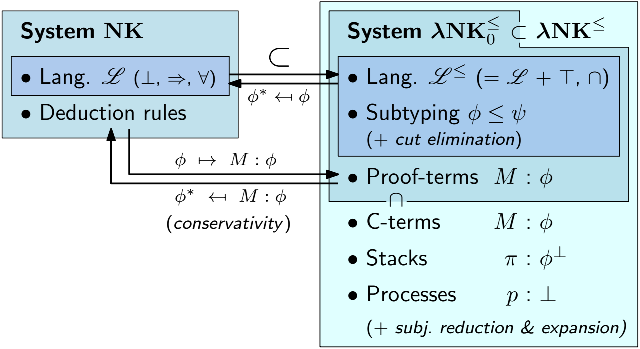
$$\frac{\Gamma, \xi : \varphi \vdash M : \psi}{\Gamma \vdash \lambda\xi \,.\, M : \varphi \Rightarrow \psi}\ (\textsc{Lam}) \qquad \frac{\Gamma \vdash M : \varphi \Rightarrow \psi \quad \Gamma \vdash N : \varphi}{\Gamma \vdash MN : \psi}\ (\textsc{App})$$

$$\frac{\Gamma, \xi : \varphi \Rightarrow \psi \vdash M : \varphi}{\Gamma \vdash \mathcal{C}\xi \,.\, M : \varphi}\ (\textsc{Call/cc})$$

$$\frac{}{\Gamma \vdash M : \top}\ (\top_{\mathrm{I}}) \quad \text{(if } FV(M) \subseteq \mathrm{dom}(\Gamma)\text{)} \qquad \frac{\Gamma \vdash M : \varphi_1 \quad \Gamma \vdash M : \varphi_2}{\Gamma \vdash M : \varphi_1 \cap \varphi_2}\ (\cap_{\mathrm{I}})$$

$$\frac{\Gamma \vdash M : \varphi}{\Gamma \vdash M : \forall x\, \varphi}\ (\forall_{\mathrm{I}}) \quad \text{(if } x \notin FV(\Gamma)\text{)} \qquad \frac{\Gamma \vdash M : \varphi \quad \varphi \leq \varphi'}{\Gamma \vdash M : \varphi'}\ (\leq)$$

Sketch of the proof of strong completeness (5/11)



$$\phi, \psi \in \mathscr{L}^{\leq} \quad ::= \quad \perp \quad | \quad \top \quad | \quad p(t_1, \ldots, t_k) \quad | \quad \phi \cap \psi \quad | \quad \phi \Rightarrow \psi \quad | \quad \forall x\, \phi$$

$$M, N \quad ::= \quad \xi \quad | \quad \maltese_\phi \quad | \quad \lambda\xi\,.\,M \quad | \quad M\,N \quad | \quad \mathcal{C}\xi\,.\,M \quad | \quad \langle\pi\rangle \qquad (\phi \text{ closed})$$

$$\pi, \pi' \quad ::= \quad \alpha_\phi \quad | \quad M \cdot \pi \qquad (\phi \equiv \perp \text{ or } \phi \equiv p(\vec{t}) \text{ closed})$$

$$p, q \quad ::= \quad M \star \pi$$

Sketch of the proof of strong completeness (6/11)

Typing rules for c-terms (⊃ proof terms)

$$\frac{}{\Gamma \vdash \xi : \varphi}\ (\textsc{Var})\quad \text{(if } (\xi : \varphi) \in \Gamma\text{)} \qquad \frac{}{\Gamma \vdash \maltese_\varphi : \varphi}\ (\textsc{Dai})\quad (\varphi \text{ closed})$$

$$\frac{\Gamma, \xi : \varphi \vdash M : \psi}{\Gamma \vdash \lambda\xi\,.\,M : \varphi \Rightarrow \psi}\ (\textsc{Lam}) \qquad \frac{\Gamma \vdash M : \varphi \Rightarrow \psi \quad \Gamma \vdash N : \varphi}{\Gamma \vdash MN : \psi}\ (\textsc{App})$$

$$\frac{\Gamma, \xi : \varphi \Rightarrow \psi \vdash M : \varphi}{\Gamma \vdash \mathcal{C}\xi\,.\,M : \varphi}\ (\textsc{Call/cc}) \qquad \frac{\Gamma \vdash \pi : \varphi^{\perp}}{\Gamma \vdash \langle\pi\rangle : \varphi \Rightarrow \psi}\ (\textsc{Cont})$$

$$\frac{}{\Gamma \vdash M : \top}\ (\top_{\mathrm{I}})\quad \text{(if } FV(M) \subseteq \mathrm{dom}(\Gamma)\text{)} \qquad \frac{\Gamma \vdash M : \varphi_1 \quad \Gamma \vdash M : \varphi_2}{\Gamma \vdash M : \varphi_1 \cap \varphi_2}\ (\cap_{\mathrm{I}})$$

$$\frac{\Gamma \vdash M : \varphi}{\Gamma \vdash M : \forall x\,\varphi}\ (\forall_{\mathrm{I}})\quad \text{(if } x \notin FV(\Gamma)\text{)} \qquad \frac{\Gamma \vdash M : \varphi \quad \varphi \leq \varphi'}{\Gamma \vdash M : \varphi'}\ (\leq)$$

Sketch of the proof of strong completeness (7/11)

Typing rules for stacks

$$\frac{}{\Gamma \vdash \alpha_\varphi : \varphi^\perp}\ (\textsc{Nil}) \quad (\varphi \equiv \bot \text{ or } \varphi \equiv p(\vec{t}) \text{ closed})$$

$$\frac{\Gamma \vdash M : \varphi \quad \Gamma \vdash \pi : \psi^\perp}{\Gamma \vdash M \cdot \pi : (\varphi \Rightarrow \psi)^\perp}\ (\textsc{Cons}) \qquad \frac{\Gamma \vdash \pi : \varphi^\perp \quad \varphi' \leq \varphi}{\Gamma \vdash \pi : \varphi'^\perp}\ (\leq^\perp)$$

Typing rule for processes

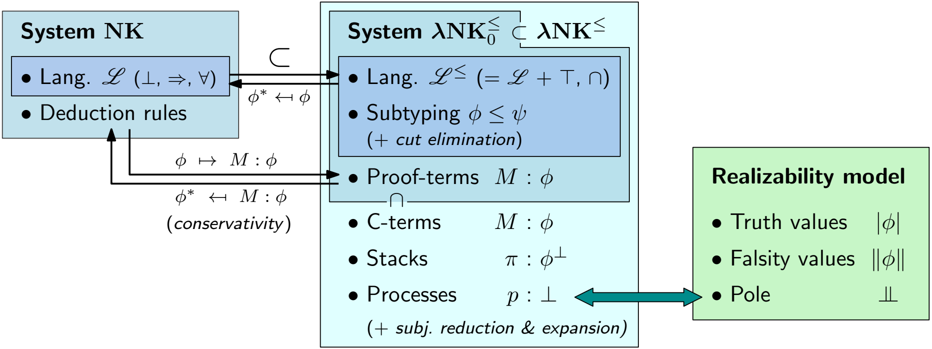
$$\frac{\Gamma \vdash M : \varphi \qquad \Gamma \vdash \pi : \varphi^\perp}{\Gamma \vdash M \star \pi : \bot}$$

Theorem

System $\boldsymbol{\lambda}\mathbf{NK}^{\leq}$ enjoys both subject reduction and subject expansion, for c-terms, stacks and processes

- C-terms and stacks are only equipped with β-reduction
- Processes are equipped with a richer (i.e. Krivine style) notion of evaluation

Sketch of the proof of strong completeness (8/11)



$$
\begin{array}{rcl}
\phi, \psi \in \mathscr{L}^{\leq} & ::= & \bot \mid \top \mid p(t_1, \ldots, t_k) \mid \phi \cap \psi \mid \phi \Rightarrow \psi \mid \forall x\, \phi \\
M, N & ::= & \xi \mid \maltese_\phi \mid \lambda \xi . M \mid M\, N \mid \mathcal{C}\xi . M \mid \langle \pi \rangle \qquad (\phi \text{ closed}) \\
\pi, \pi' & ::= & \alpha_\phi \mid M \cdot \pi \qquad (\phi \equiv \bot \text{ or } \phi \equiv p(\vec{t}) \text{ closed}) \\
p, q & ::= & M \star \pi
\end{array}
$$

Sketch of the proof of strong completeness (9/11)

Pole & domain of the model

$$\begin{array}{rcl} \perp\!\!\!\perp & = & \text{set of closed well-typed processes } (\vdash p : \bot) \\ \mathscr{M} & = & \text{set of closed 1st-order terms (Herbrand universe)} \end{array}$$

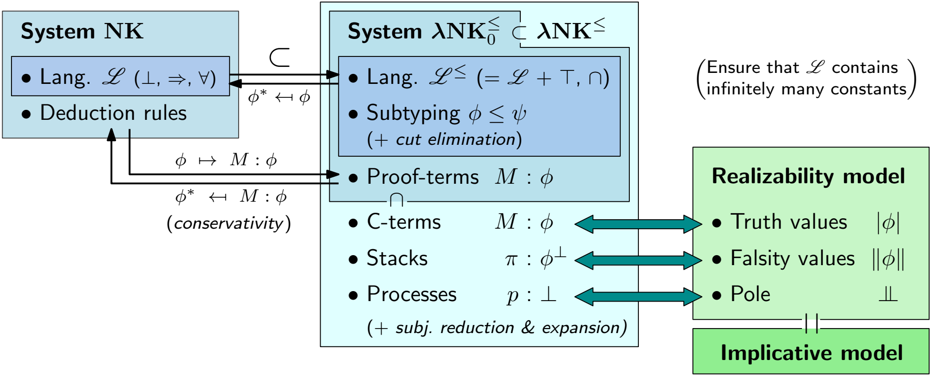
Falsity value $\|\varphi\|$ ($\subseteq \Pi$) of a closed formula φ

$$\begin{array}{rcl} \|\bot\| & = & \Pi \\ \|\top\| & = & \varnothing \\ \|p(t_1,\ldots,t_k)\| & = & \{\alpha_{p(t_1,\ldots,t_k)}\} \\ \|\varphi \cap \psi\| & = & \|\varphi\| \cup \|\psi\| \\ \|\varphi \Rightarrow \psi\| & = & \{M \cdot \pi \mid M \in |\varphi|,\ \pi \in \|\psi\|\} \\ \|\forall x\, \varphi\| & = & \displaystyle\bigcup_{t \in \mathscr{M}} \|\varphi[x := t]\| \end{array}$$

Truth value $|\varphi|$ ($\subseteq \Lambda$) of a closed formula φ

$$|\varphi| \;=\; \|\varphi\|^{\perp\!\!\!\perp} \;=\; \{M \in \Lambda \mid \forall \pi \in \|\varphi\|,\ M \star \pi \in \perp\!\!\!\perp\}$$

Sketch of the proof of strong completeness (10/11)



Sketch of the proof of strong completeness (11/11)

- The above proof sketch shows how any 1st-order language $\mathscr{L}$ can be turned into an implicative model $\mathscr{M}$ such that

$$\vdash_{\mathsf{LK}} \varphi \qquad \text{iff} \qquad [\![\varphi]\!]^{\mathscr{M}} \in S_K^0(\mathscr{A})$$

  Note that:
  - The implicative structure $\mathscr{A}$ underlying $\mathscr{M}$ is actually derived from the (sets of) stacks of system $\boldsymbol{\lambda}\mathbf{NK}^{\leq}$
  - The underlying separator $S_K^0(\mathscr{A})$ (classical core of $\mathscr{A}$) is generated from the proof terms of system $\boldsymbol{\lambda}\mathbf{NK}^{\leq}$ (due to conservativity)
  - The $\mathscr{A}$-model $\mathscr{M}$ is the Herbrand universe of language $\mathscr{L}$ (assuming that $\mathscr{L}$ has infinitely many constants)

- To incorporate the axioms of the 1st-order theory $\mathscr{T}$, just take the separator $S \subseteq \mathscr{A}$ generated from the corresponding denotations

- **Morality:** Implicative algebra =
  1st-order theory expressed into semantics terms

Deducing first-order completeness

Let $\mathscr{T}$ be a consistent (classical) 1st-order theory

- From the strong completeness theorem, there is a full implicative model $\mathscr{M}$ (over some classical implicative algebra $\mathscr{A}$) such that:

  $$\mathscr{T} \vdash \varphi \quad \text{iff} \quad \mathscr{M} \models \varphi \qquad (\varphi \text{ closed})$$

  Moreover the implicative algebra $\mathscr{A}$ is consistent since the theory $\mathscr{T}$ is

- Picking some ultraseparator $U \supseteq S_{\mathscr{A}}$, get a Tarski model $\mathscr{M} : U$:

  $$\mathscr{T} \vdash \varphi \quad \text{implies} \quad \mathscr{M}{:}U \models \varphi \qquad (\varphi \text{ closed})$$

Therefore we get:

Factorization of 1st-order completeness

FO-theory		**Impl. model**		**Tarski model**
$\mathscr{T} \vdash \varphi$	$\Longleftrightarrow$	$\mathscr{M} \models \varphi$	$\overset{U}{\Longrightarrow}$	$\mathscr{M}{:}U \models \varphi$
$\exists t \in \mathsf{PL},\ t : \varphi$	$\Longleftrightarrow$	$\exists t' \in \mathsf{PL},\ t' \Vdash \varphi$	$\Longrightarrow$	$[\![\varphi]\!]^{\mathscr{M}} \in U$
	(constructive)		(non constr.)	

Conclusion & work in progress

- Implicative algebra: a simple structure to factorize model-theoretic constructions underlying forcing and realizability (both in LJ and LK)

- **Higher-order completeness** (recall): each $\mathbf{Set}$-based tripos is isomorphic to some implicative tripos

- **First-order completeness** (this talk): each classical first-order theory $\mathscr{T}$ is captured by some classical implicative model $\mathscr{M}$:

  $$\mathscr{T} \vdash \varphi \qquad \text{iff} \qquad \mathscr{M} \models \varphi \qquad\qquad \text{(strong completeness)}$$

  Result is not new — cf Boolean-valued models — but proof technique is

- Unlike the Boolean-valued model construction, our construction preserves the computational contents of proofs

  Conjecture: This construction should work for intuitionistic theories too

- Since 1st-order hyperdoctrines $\cong$ multi-sorted 1st-order theories, can we deduce that *"all 1st-order hyperdoctrines are implicative"* ?