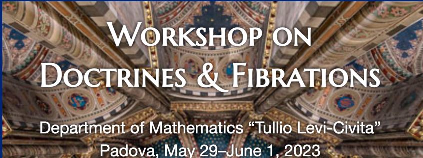
Università
di Genova

DIMA DIPARTIMENTO
DI MATEMATICA

Hyperdoctrines and logic
a tutorial

Giuseppe Rosolini

with a lot of help from many friends



WORKSHOP ON
DOCTRINES & FIBRATIONS

Department of Mathematics “Tullio Levi-Civita”
Padova, May 29–June 1, 2023

Doctrines and hyperdoctrines

Logic in categories	monoidal closed category cartesian closed category Grothendieck topos exact category elementary topos pretopos Heyting pretopos arithmetic universe logos...
Logic in category theory	hyperdoctrine existential elementary doctrine tripos indexed poset... Grothendieck fibration elementary fibration...

F.W. Lawvere

Ordinal sums and equational doctrines *Sem. on Triples and Cat. Homology Theory* 1966/67

Adjontness in foundations *Dialectica* 1969, available also in *Repr. Theory Appl. Categ.*

Doctrines and hyperdoctrines

Logic in categories

Sub: $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{P}os$

monoidal closed category
cartesian closed category
Grothendieck topos
exact category
elementary topos
pretopos
Heyting pretopos
arithmetic universe
logos...

Logic in category theory

P: $\mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{P}os$

hyperdoctrine
existential elementary doctrine
tripos
indexed poset...
Grothendieck fibration
elementary fibration...

F.W. Lawvere
Ordinal sums and equational doctrines *Sem. on Triples and Cat. Homology Theory* 1966/67
Adjontness in foundations *Dialectica* 1969, available also in *Repr. Theory Appl. Categ.*

The doctrine of indexed posets

$$\begin{tikzcd}
\mathcal{I}dx\mathcal{P}os \arrow[r, hook, shift right=2, "\text{full}"'] \arrow[r, phantom, "\perp"] & \mathcal{I}dx\mathcal{C}at \equiv \mathcal{F}ib \arrow[l, bend right, shift right=2] \arrow[r, shift right=2] \arrow[r, phantom, "\perp"] & \mathcal{C}at^{\to} \arrow[l, bend right, shift right=2]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B} \arrow[d, shift right=3, "\overline{f}"'] \arrow[d, shift left=3, "\overline{g}"] \arrow[d, phantom, "\xrightarrow[\overline{\theta}]{\bullet}"] \\
\mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}^{\mathrm{op}} \arrow[dd, shift right=3, "\overline{f}^{\mathrm{op}}"'] \arrow[dd, shift left=3, "\overline{g}^{\mathrm{op}}"] \arrow[dd, phantom, "\xleftarrow[\overline{\theta}^{\mathrm{op}}]{\bullet}"] \arrow[rd, "P"] & \\
{} \arrow[r, phantom, "\hat{f}\downarrow\bullet\leq\bullet\downarrow\hat{g}"] & \mathcal{P}os \\
\mathcal{C}^{\mathrm{op}} \arrow[ru, "Q"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\int P \arrow[r, shift left=2, "\hat{f}"] \arrow[r, shift right=2, "\hat{g}"'] \arrow[r, phantom, "\hat{\theta}\downarrow\bullet"] \arrow[d, "p"', "\text{fib}" near end] & \int Q \arrow[d, "q", "\text{fib}"' near end] \\
\mathcal{B} \arrow[r, shift left=2, "\overline{f}"] \arrow[r, shift right=2, "\overline{g}"'] \arrow[r, phantom, "\overline{\theta}\downarrow\bullet"] & \mathcal{C}
\end{tikzcd}$$

The doctrine of indexed posets

$$\begin{tikzcd} \mathcal{IdxPos} \arrow[r, hook, "\mathrm{full}"'] \arrow[r, phantom, shift left=3, "\perp"] & \mathcal{IdxCat} \arrow[l, bend right, shift right=2] \equiv \mathcal{Fib} \arrow[r] \arrow[r, phantom, shift left=3, "\perp"] & \mathcal{Cat}^{\rightarrow} \arrow[l, bend right, shift right=2] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{B} \arrow[d, shift right=3, "\bar f"'] \arrow[d, shift left=3, "\bar g"] \arrow[d, phantom, "\overset{\bullet}{\rightarrow}\bar\theta"] \\ \mathcal{C} \end{tikzcd}
\qquad
\begin{tikzcd} \mathcal{B}^{\mathrm{op}} \arrow[dd, shift right=3, "\bar f^{\mathrm{op}}"'] \arrow[dd, shift left=3, "\bar g^{\mathrm{op}}"] \arrow[dd, phantom, "\overset{\bullet}{\leftarrow}\bar\theta^{\mathrm{op}}"] \arrow[dr, "P"] & \\ & \mathcal{Pos} \arrow[l, phantom, "\hat f \downarrow {\bullet}{\leq}{\bullet} \downarrow \hat g"] \\ \mathcal{C}^{\mathrm{op}} \arrow[ur, "Q"'] & \end{tikzcd}
\qquad
\begin{tikzcd} \int P \arrow[r, shift left=2, "\hat f"] \arrow[r, shift right=2, "\hat g"'] \arrow[r, phantom, "\hat\theta\downarrow\bullet"] \arrow[d, "p"'] \arrow[d, "\mathrm{fib}"', pos=0.9] & \int Q \arrow[d, "q"] \arrow[d, "\mathrm{fib}", pos=0.9] \\ \mathcal{B} \arrow[r, shift left=2, "\bar f"] \arrow[r, shift right=2, "\bar g"'] \arrow[r, phantom, "\bar\theta\downarrow\bullet"] & \mathcal{C} \end{tikzcd}$$

Proposition

There is a lax idempotent 2-monad on $\mathcal{Cat}^{\rightarrow}$ *whose algebras are indexed posets*

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a ***primary doctrine*** when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets

$$\begin{tikzcd} P(b)\times P(b) \arrow[r, shift right=1.5ex, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, shift right=1.5ex, "{\langle \mathrm{id}, \mathrm{id}\rangle}"'] \arrow[r, shift left=1.5ex, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, shift left=1.5ex, "\top_b"] \end{tikzcd}$$

- $(\wedge_b)_b \colon P \times P \stackrel{\cdot}{\longrightarrow} P$ and $(\top_b)_b \colon 1 \stackrel{\cdot}{\longrightarrow} P$

F.W. Lawvere

Equality in hyperdoctrines and comprehension schema as an adjoint functor

Applications of Categorical Algebra 1970

A. Pitts

Categorical Logic *Handbook of Logic in Computer Science* 1995

B. Jacobs

Categorical Logic and Type Theory 1999

M.E. Maietti, G. R.

Quotient completion for the foundation of constructive mathematics

Logica Universalis 2013

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a ***primary doctrine*** when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets $$\begin{tikzcd} P(b) \times P(b) \arrow[r, bend right, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, bend right, "\langle \mathrm{id}, \mathrm{id} \rangle"'] \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, bend left, "\top_b"] \end{tikzcd}$$
- $P(f)(\alpha \wedge_c \beta) = P(f)(\alpha) \wedge_b P(f)(\beta)$ and $P(f)(\top_c) = \top_b$ for $f\colon b \longrightarrow c$ in $\mathcal{B}$

F.W. Lawvere

Equality in hyperdoctrines and comprehension schema as an adjoint functor
Applications of Categorical Algebra 1970

A. Pitts

Categorical Logic *Handbook of Logic in Computer Science* 1995

B. Jacobs

Categorical Logic and Type Theory 1999

M.E. Maietti, G. R.

Quotient completion for the foundation of constructive mathematics
Logica Universalis 2013

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a ***primary doctrine*** when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets
$$\begin{tikzcd} P(b) \times P(b) \arrow[r, shift right=1.5ex, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, shift right=1.5ex, "\langle \mathrm{id}, \mathrm{id} \rangle"'] \arrow[r, shift left=1.5ex, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, shift left=1.5ex, "\top_b"] \end{tikzcd}$$
- $P(f)(\alpha \wedge_c \beta) = P(f)(\alpha) \wedge_b P(f)(\beta)$ and $P(f)(\top_c) = \top_b$ for $f\colon b \xrightarrow{} c$ in $\mathcal{B}$

Examples

$\mathcal{T}$: a theory in the $\wedge$-fragment of first order logic

$\mathcal{C}tx_{\mathcal{L}}$: $(x_{i_1}, \dots, x_{i_n}) \xrightarrow{(t_1, \dots, t_m)} (x_{j_1}, \dots, x_{j_m})$

$\mathrm{LT}^{\mathcal{T}}_{\wedge}(x_{i_1}, \dots, x_{i_n})$: $\phi(x_{i_1}, \dots, x_{i_n}) \vdash_{\mathcal{T}} \psi(x_{i_1}, \dots, x_{i_n})$

$$\mathrm{LT}^{\mathcal{T}}_{\wedge}(\vec{t})(\phi(\vec{x}) \wedge \psi(\vec{x})) = (\phi(\vec{x}) \wedge \psi(\vec{x}))[\vec{t}/\vec{x}] = \phi[\vec{t}/\vec{x}] \wedge \psi[\vec{t}/\vec{x}] = \mathrm{LT}^{\mathcal{T}}_{\wedge}(\vec{t})(\phi(\vec{x})) \wedge \mathrm{LT}^{\mathcal{T}}_{\wedge}(\vec{t})(\psi(\vec{x}))$$

$\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}^{\mathcal{T}}_{\wedge}} \mathcal{P}os$ is a primary doctrine

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a ***primary doctrine*** when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets $$\begin{tikzcd} P(b) \times P(b) \arrow[r, shift right, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, shift right, "\langle \mathrm{id}, \mathrm{id} \rangle"'] \arrow[r, shift left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, shift left, "\top_b"] \end{tikzcd}$$
- $(\alpha \wedge_c \beta)(f) = \alpha(f) \wedge_b \beta(f)$ and $\top_c(f) = \top_b$ for $f \colon b \longrightarrow c$ in $\mathcal{B}$

Examples

$\mathcal{T}$: a theory in the $\wedge$-fragment of first order logic

$\mathcal{C}tx_{\mathcal{L}}$: $$\begin{tikzcd} (x_{i_1}, \dots, x_{i_n}) \arrow[r, bend left, "(t_1, \dots, t_m)"'] & (x_{j_1}, \dots, x_{j_m}) \end{tikzcd}$$

$\mathrm{LT}^{\mathcal{T}}_{\wedge}(x_{i_1}, \dots, x_{i_n})$: $\phi(x_{i_1}, \dots, x_{i_n}) \vdash_{\mathcal{T}} \psi(x_{i_1}, \dots, x_{i_n})$

$$\mathrm{LT}^{\mathcal{T}}_{\wedge}(\vec{t})(\phi(\vec{x}) \wedge \psi(\vec{x})) = (\phi(\vec{x}) \wedge \psi(\vec{x}))[\vec{t}/\vec{x}] = \phi[\vec{t}/\vec{x}] \wedge \psi[\vec{t}/\vec{x}] = \mathrm{LT}^{\mathcal{T}}_{\wedge}(\vec{t})(\phi(\vec{x})) \wedge \mathrm{LT}^{\mathcal{T}}_{\wedge}(\vec{t})(\psi(\vec{x}))$$

$\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}^{\mathcal{T}}_{\wedge}} \mathcal{P}os$ is a primary doctrine

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a ***primary doctrine*** when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets $$\begin{tikzcd} P(b)\times P(b) \arrow[r, shift right=2, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, shift right=2, "\langle \mathrm{id}, \mathrm{id}\rangle"'] \arrow[r, shift left=2, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, shift left=2, "\top_b"] \end{tikzcd}$$
- $(\alpha \wedge_c \beta)(f) = \alpha(f) \wedge_b \beta(f)$ and $\top_c(f) = \top_b$ for $f\colon b \xrightarrow{} c$ in $\mathcal{B}$

Examples

The powerset functor $\mathcal{S}et^{\mathrm{op}} \xrightarrow{\mathbf{P}} \mathcal{P}os$

The subobject functor $\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{P}os$ is primary when $\mathcal{C}$ has finite limits

$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathcal{M}} \mathcal{P}os$ is primary for $\mathcal{C}$ with finite products, and
$\mathcal{M}$ a class of monos in $\mathcal{C}$
closed under isos and pullbacks

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a ***primary doctrine*** when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets $$\begin{tikzcd} P(b) \times P(b) \arrow[r, shift right=2, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, shift right=2, "\langle \mathrm{id}, \mathrm{id} \rangle"'] \arrow[r, shift left=2, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, shift left=2, "\top_b"] \end{tikzcd}$$
- $f^{-1}(\alpha \wedge_c \beta) = f^{-1}(\alpha) \wedge_b f^{-1}(\beta)$ and $f^{-1}(\top_c) = \top_b$ for $f\colon b \xrightarrow{} c$ in $\mathcal{B}$

Examples

The powerset functor $\mathcal{S}et^{\mathrm{op}} \xrightarrow{\mathbf{P}} \mathcal{P}os$

The subobject functor $\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{P}os$ is primary when $\mathcal{C}$ has finite limits

$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathcal{M}} \mathcal{P}os$ is primary for $\mathcal{C}$ with finite products, and
$\mathcal{M}$ a class of monos in $\mathcal{C}$
closed under isos and pullbacks

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a ***primary doctrine*** when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets $$\begin{tikzcd} P(b)\times P(b) \arrow[r, shift right=1.5ex, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, shift right=1.5ex, "\langle \mathrm{id}, \mathrm{id}\rangle"'] \arrow[r, shift left=1.5ex, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, shift left=1.5ex, "\top_b"] \end{tikzcd}$$
- $f^{-1}(\alpha \wedge_c \beta) = f^{-1}(\alpha) \wedge_b f^{-1}(\beta)$ and $f^{-1}(\top_c) = \top_b$ for $f\colon b \xrightarrow{} c$ in $\mathcal{B}$

Examples

$\mathcal{A}$: a category with weak pullbacks

$\mathrm{Vrn}(a) \overset{\mathrm{df}}{=} (\mathcal{A}/a)_{\mathrm{po}}$

$$\begin{tikzcd} a_1 \arrow[rr] \arrow[dr, "g"'] & & a_2 \arrow[dl, "h"] \\ & a \arrow[u, phantom, "\leq"] & \end{tikzcd}$$

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a ***primary doctrine*** when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets $$\begin{tikzcd}[column sep=large] P(b)\times P(b) \arrow[r, shift right, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, shift right, "\langle \mathrm{id}, \mathrm{id}\rangle"'] \arrow[r, shift left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, shift left, "\top_b"] \end{tikzcd}$$
- $f^{-1}(\alpha \wedge_c \beta) = f^{-1}(\alpha) \wedge_b f^{-1}(\beta)$ and $f^{-1}(\top_c) = \top_b$ for $f\colon b \longrightarrow c$ in $\mathcal{B}$

Examples

$\mathcal{A}$: a category with weak pullbacks

$$\begin{tikzcd} q \arrow[r] \arrow[d, "g'"'] \arrow[dr, phantom, "\text{w.p.}"] & a_1 \arrow[d, "g"] \\ b \arrow[r, "f"] & a \end{tikzcd}$$

$\mathrm{Vrn}(a) \overset{\mathrm{df}}{=} (\mathcal{A}/a)_{\mathrm{po}}$

$\mathrm{Vrn}(f)\colon [g]_{\lesseqgtr} \mapsto [g']_{\lesseqgtr}\colon \mathrm{Vrn}(a) \longrightarrow \mathrm{Vrn}(b)$

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a ***primary doctrine*** when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets

$$\begin{tikzcd} P(b) \times P(b) \arrow[r, bend right, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, bend right, "\langle \mathrm{id}, \mathrm{id} \rangle"'] \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, bend left, "\top_b"] \end{tikzcd}$$

- $f^{-1}(\alpha \wedge_c \beta) = f^{-1}(\alpha) \wedge_b f^{-1}(\beta)$ and $f^{-1}(\top_c) = \top_b$ for $f\colon b \longrightarrow c$ in $\mathcal{B}$

Examples

$\mathcal{A}$: a category with weak pullbacks

$$\begin{tikzcd} q \arrow[r] \arrow[d, "g'"'] \arrow[dr, phantom, "\text{w.p.}"] & a_1 \arrow[d, "g"] \\ b \arrow[r, "f"] & a \end{tikzcd}$$

$\mathrm{Vrn}(a) \overset{\mathrm{df}}{=} (\mathcal{A}/a)_{\mathrm{po}}$ $\qquad$ $\mathrm{Vrn}(f)\colon [g]_{\lesseqgtr} \mapsto [g']_{\lesseqgtr} \colon \mathrm{Vrn}(a) \longrightarrow \mathrm{Vrn}(b)$

$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Vrn}} \mathcal{P}os$ is primary when $\mathcal{A}$ has (strong) finite products

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a ***primary doctrine*** when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets $$\begin{tikzcd} P(b) \times P(b) \arrow[r, shift right, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, shift right, "\langle \mathrm{id}, \mathrm{id} \rangle"'] \arrow[r, shift left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, shift left, "\top_b"] \end{tikzcd}$$
- $f^*(\alpha \wedge_c \beta) = f^*(\alpha) \wedge_b f^*(\beta)$ and $f^*(\top_c) = \top_b$ for $f\colon b \longrightarrow c$ in $\mathcal{B}$

Examples

$\mathcal{A}$: a category with weak pullbacks

$$\begin{tikzcd} q \arrow[r] \arrow[d, "g'"'] \arrow[dr, phantom, "\text{w.p.}"] & a_1 \arrow[d, "g"] \\ b \arrow[r, "f"] & a \end{tikzcd}$$

$\mathrm{Vrn}(a) \overset{\mathrm{df}}{=} (\mathcal{A}/a)_{\mathrm{po}}$ $\qquad$ $\mathrm{Vrn}(f)\colon [g]_{\lesseqgtr} \mapsto [g']_{\lesseqgtr} \colon \mathrm{Vrn}(a) \longrightarrow \mathrm{Vrn}(b)$

$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Vrn}} \mathcal{P}os$ is primary when $\mathcal{A}$ has (strong) finite products

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{Pos}$ is a ***primary doctrine*** when

- ▶ $\mathcal{B}$ has finite products
- ▶ each fibre $P(b)$ has finite meets $$\begin{tikzcd} P(b) \times P(b) \arrow[r, shift right, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, shift right, "{\langle \mathrm{id}, \mathrm{id} \rangle}"'] \arrow[r, shift left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, shift left, "\top_b"] \end{tikzcd}$$
- ▶ $f^*(\alpha \wedge_c \beta) = f^*(\alpha) \wedge_b f^*(\beta)$ and $f^*(\top_c) = \top_b$ for $f \colon b \xrightarrow{} c$ in $\mathcal{B}$

Examples

H: an inf-semilattice

$$\begin{tikzcd} \mathcal{Set}^{\mathrm{op}} \arrow[rr, "H^{(-)}"] && \mathcal{Pos} \\ S \arrow[rr, mapsto] \arrow[dd, "f"'] && H^S \arrow[dd, "- \circ f"] \\ {} \arrow[rr, mapsto] && {} \\ S' \arrow[rr, mapsto] && H^{S'} \end{tikzcd}$$

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a ***primary doctrine*** when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets

$$\begin{tikzcd}
P(b) \times P(b) \arrow[r, shift right=2, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, shift right=2, "{\langle \mathrm{id}, \mathrm{id} \rangle}"'] \arrow[r, shift left=2, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, shift left=2, "\top_b"]
\end{tikzcd}$$

- $(\alpha \wedge_c \beta)(f) = \alpha(f) \wedge_b \beta(f)$ and $\top_c(f) = \top_b$ for $f\colon b \xrightarrow{} c$ in $\mathcal{B}$

Examples

H: an inf-semilattice

$$\begin{tikzcd}
\mathcal{S}et^{\mathrm{op}} \arrow[r, "H^{(-)}"] & \mathcal{P}os \\
S \arrow[r, mapsto] \arrow[d, "f"', ""{name=F}] & H^{S} \arrow[d, "- \circ f", ""'{name=G}] \\
S' \arrow[r, mapsto] & H^{S'}
\arrow[mapsto, from=F, to=G]
\end{tikzcd}$$

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a ***primary doctrine*** when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets

$$\begin{tikzcd}
P(b) \times P(b) \arrow[r, bend right, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, bend right, "\langle \mathrm{id}, \mathrm{id} \rangle"'] \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, bend left, "\top_b"]
\end{tikzcd}$$

- $(\alpha \wedge_c \beta)f = \alpha f \wedge_b \beta f$ and $\top_c f = \top_b$ for $f\colon b \xrightarrow{} c$ in $\mathcal{B}$

Examples

H: an inf-semilattice

$$\begin{tikzcd}
\mathcal{S}et^{\mathrm{op}} \arrow[rr, "H^{(-)}"] & & \mathcal{P}os \\
S \arrow[rr, mapsto] \arrow[d, "f"'] & & H^S \arrow[d, "- \circ f"] \\
S' \arrow[rr, mapsto] & & H^{S'}
\end{tikzcd}$$

$$\begin{tikzcd}
f \arrow[r, mapsto] & - \circ f
\end{tikzcd}$$

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a ***primary doctrine*** when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets $$\begin{tikzcd} P(b) \times P(b) \arrow[r, shift right=2, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, shift right=2, "\langle \mathrm{id}, \mathrm{id} \rangle"'] \arrow[r, shift left=2, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, shift left=2, "\top_b"] \end{tikzcd}$$
- $(\alpha \wedge_c \beta)f = \alpha f \wedge_b \beta f$ and $\top_c f = \top_b$ for $f \colon b \longrightarrow c$ in $\mathcal{B}$

Proposition

$$\begin{tikzcd} \mathcal{I}dx\mathcal{P}os(\mathcal{B}) \arrow[r, equal] & \mathrm{Pos}([\mathcal{B}^{\mathrm{op}}, \mathcal{S}et]) \\ \mathcal{P}rim\mathcal{D}tn(\mathcal{B}) \arrow[r, equal] \arrow[u, "U"] & \mathrm{InfSLatt}([\mathcal{B}^{\mathrm{op}}, \mathcal{S}et]) \arrow[u, "U"'] \end{tikzcd}$$

Primary doctrines

Definition

An indexed poset $\mathcal{B}^{\mathrm{op}} \xrightarrow{P} \mathcal{P}os$ is a ***primary doctrine*** when

- $\mathcal{B}$ has finite products
- each fibre $P(b)$ has finite meets

$$\begin{tikzcd} P(b) \times P(b) \arrow[r, shift right, "\wedge_b"'] \arrow[r, phantom, "\perp"] & P(b) \arrow[l, shift right, "\langle \mathrm{id}, \mathrm{id} \rangle"'] \arrow[r, shift left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, shift left, "\top_b"] \end{tikzcd}$$

- $(\alpha \wedge_c \beta){\upharpoonright_f} = \alpha {\upharpoonright_f} \wedge_b \beta {\upharpoonright_f}$ and $\top_c {\upharpoonright_f} = \top_b$ for $f\colon b \longrightarrow c$ in $\mathcal{B}$

Proposition

$$\begin{tikzcd} \mathcal{I}dx\mathcal{P}os(\mathcal{B}) \arrow[r, equal] & \mathrm{Pos}([\mathcal{B}^{\mathrm{op}}, \mathcal{S}et]) \\ \mathcal{P}rim\mathcal{D}tn(\mathcal{B}) \arrow[r, equal] \arrow[u, "U"] & \mathrm{InfSLatt}([\mathcal{B}^{\mathrm{op}}, \mathcal{S}et]) \arrow[u, "U"'] \end{tikzcd}$$

Existential doctrines

Definition

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ is ***existential*** when

- reindexings along projections $\mathrm{p}_1 \colon b \times c \longrightarrow b$ have left adjoints

$$\begin{tikzcd} D(b \times c) \arrow[r, bend left, "\exists^{\mathrm{p}_1}"] \arrow[r, phantom, "\perp"] & D(b) \arrow[l, bend left, "D(\mathrm{p}_1)"] \end{tikzcd}$$

- each left adjoint satisfies FR: $\exists^{\mathrm{p}_1}\left(D(\mathrm{p}_1)(\alpha) \wedge \beta\right) = \alpha \wedge \exists^{\mathrm{p}_1}(\beta)$
- $\left(\exists^{\mathrm{p}_1 : b \times c \to b}\right)_b : D(- \times c) \stackrel{\cdot}{\longrightarrow} D$

Existential doctrines

Definition

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ is ***existential*** when

- reindexings along projections $\mathrm{p}_1 \colon b \times c \longrightarrow b$ have left adjoints

$$\begin{tikzcd}
D(b \times c) \arrow[r, shift left=2, "\exists^{\mathrm{p}_1}"] \arrow[r, phantom, "\perp"] & D(b) \arrow[l, shift left=2, "D(\mathrm{p}_1)"]
\end{tikzcd}$$

- each left adjoint satisfies FR: $\exists^{\mathrm{p}_1}\big(D(\mathrm{p}_1)(\alpha) \wedge \beta\big) = \alpha \wedge \exists^{\mathrm{p}_1}(\beta)$
- $D(f)\big(\exists^{\mathrm{p}_1 : b \times c \to b}(\beta)\big) = \exists^{\mathrm{p}'_1 : b' \times c \to b'}\big(D(f \times \mathrm{id}_c)(\beta)\big)$ for $f \colon b' \longrightarrow b$ in $\mathcal{B}$

$$\begin{tikzcd}
D(b \times c) \arrow[r, "\exists^{\mathrm{p}_1}"] \arrow[d, "D(f \times \mathrm{id})"'] & D(b') \arrow[d, "D(f)"] \\
D(b' \times c) \arrow[r, "\exists^{\mathrm{p}'_1}"'] & D(b')
\end{tikzcd}$$

Existential doctrines

Definition

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ is ***existential*** when

- reindexings along projections $\mathrm{p}_1 \colon b \times c \longrightarrow b$ have left adjoints

$$\begin{tikzcd}
D(b \times c) \arrow[r, shift left=1.5ex, "\exists^{\mathrm{p}_1}"] \arrow[r, phantom, "\perp"] & D(b) \arrow[l, shift left=1.5ex, "D(\mathrm{p}_1)"]
\end{tikzcd}$$

- each left adjoint satisfies FR: $\exists^{\mathrm{p}_1}(\alpha(\mathrm{p}_1) \wedge \beta) = \alpha \wedge \exists^{\mathrm{p}_1}(\beta)$
- $D(f)\left(\exists^{\mathrm{p}_1 : b \times c \to b}(\beta)\right) = \exists^{\mathrm{p}'_1 : b' \times c \to b'}\left(D(f \times \mathrm{id}_c)(\beta)\right)$ for $f \colon b' \longrightarrow b$ in $\mathcal{B}$

$$\begin{tikzcd}
 & b \\
b \times c \arrow[ru, "\mathrm{p}_1"] \arrow[rd, "\mathrm{p}_2"'] & \\
 & c
\end{tikzcd}$$

Existential doctrines

Definition

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{Pos}$ is ***existential*** when

- reindexings along projections $\mathrm{p}_1 \colon b \times c \longrightarrow b$ have left adjoints

$$\begin{tikzcd} D(b \times c) \arrow[r, bend left, "\exists^{\mathrm{p}_1}"] \arrow[r, phantom, "\perp"] & D(b) \arrow[l, bend left, "D(\mathrm{p}_1)"] \end{tikzcd}$$

- each left adjoint satisfies FR: $\exists^{\mathrm{p}_1}(\alpha(\mathrm{p}_1) \wedge \beta) = \alpha \wedge \exists^{\mathrm{p}_1}(\beta)$
- $D(f)\left(\exists^{\mathrm{p}_1 : b \times c \to b}(\beta)\right) = \exists^{\mathrm{p}'_1 : b' \times c \to b'}\left(D(f \times \mathrm{id}_c)(\beta)\right)$ for $f \colon b' \longrightarrow b$ in $\mathcal{B}$

$$\begin{tikzcd} & b \\ b \times c \arrow[ur, "\mathrm{p}_1"] \arrow[dr, "\mathrm{p}_2"'] \arrow[r, dashed, "\langle \mathrm{p}_1, \mathrm{p}_2 \rangle"] & b \times c \arrow[u, "\mathrm{p}_1"'] \arrow[d, "\mathrm{p}_2"] \\ & c \end{tikzcd}$$

Existential doctrines

Definition

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ is ***existential*** when

- reindexings along projections $\mathrm{p}_1 : b \times c \longrightarrow b$ have left adjoints

$$\begin{tikzcd} D(b \times c) \arrow[r, shift left=2, "\exists^{\mathrm{p}_1}"] \arrow[r, phantom, "\perp"] & D(b) \arrow[l, shift left=2, "D(\mathrm{p}_1)"] \end{tikzcd}$$

- each left adjoint satisfies FR: $\exists^{\mathrm{p}_1}(\alpha(\mathrm{p}_1) \wedge \beta(\mathrm{p}_1, \mathrm{p}_2)) = \alpha \wedge \exists^{\mathrm{p}_1}(\beta(\mathrm{p}_1, \mathrm{p}_2))$
- $D(f)\left(\exists^{\mathrm{p}_1 : b \times c \to b}(\beta)\right) = \exists^{\mathrm{p}'_1 : b' \times c \to b'}\left(D(f \times \mathrm{id}_c)(\beta)\right)$ for $f : b' \longrightarrow b$ in $\mathcal{B}$

$$\begin{tikzcd} & b \\ b \times c \arrow[ur, "\mathrm{p}_1"] \arrow[r, dashed, "\langle \mathrm{p}_1, \mathrm{p}_2 \rangle"] \arrow[dr, "\mathrm{p}_2"'] & b \times c \arrow[u, "\mathrm{p}_1"'] \arrow[d, "\mathrm{p}_2"] \\ & c \end{tikzcd}$$

Existential doctrines

Definition

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ is ***existential*** when

- reindexings along projections $\mathrm{p}_1 \colon b \times c \longrightarrow b$ have left adjoints

$$\begin{tikzcd} D(b \times c) \arrow[r, shift left=2, "\exists_{\hat{2}}"] \arrow[r, phantom, "\perp"] & D(b) \arrow[l, shift left=2, "D(\mathrm{p}_1)"] \end{tikzcd}$$

- each left adjoint satisfies FR: $\exists_{\hat{2}}(\alpha(\mathrm{p}_1) \wedge \beta(\mathrm{p}_1, \mathrm{p}_2)) = \alpha \wedge \exists_{\hat{2}}(\beta(\mathrm{p}_1, \mathrm{p}_2))$
- $D(f)\left(\exists^{\mathrm{p}_1 : b \times c \to b}(\beta)\right) = \exists^{\mathrm{p}'_1 : b' \times c \to b'}\left(D(f \times \mathrm{id}_c)(\beta)\right)$ for $f \colon b' \longrightarrow b$ in $\mathcal{B}$

$$\begin{tikzcd} & & b \\ b \times c \arrow[urr, "\mathrm{p}_1"] \arrow[drr, "\mathrm{p}_2"'] \arrow[rr, dashed, "\langle \mathrm{p}_1, \mathrm{p}_2 \rangle"] & & b \times c \arrow[u, "\mathrm{p}_1"'] \arrow[d, "\mathrm{p}_2"] \\ & & c \end{tikzcd}$$

Existential doctrines

Definition

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ is ***existential*** when

- reindexings along projections $\mathrm{p}_1 \colon b \times c \longrightarrow b$ have left adjoints

$$\begin{tikzcd}
D(b \times c) \arrow[rr, bend left, "\exists_{\hat{2}}"] \arrow[rr, phantom, "\perp"] & & D(b) \arrow[ll, bend left, "D(\mathrm{p}_1)"]
\end{tikzcd}$$

- each left adjoint satisfies FR: $\exists_{\hat{2}}(\alpha(\mathrm{p}_1) \wedge \beta(\mathrm{p}_1, \mathrm{p}_2)) = \alpha(\mathrm{p}_1) \wedge \exists_{\hat{2}}(\beta(\mathrm{p}_1, \mathrm{p}_2))$
- $D(f)\left(\exists^{\mathrm{p}_1 : b \times c \to b}(\beta)\right) = \exists^{\mathrm{p}_1' : b' \times c \to b'}\left(D(f \times \mathrm{id}_c)(\beta)\right)$ for $f \colon b' \longrightarrow b$ in $\mathcal{B}$

$$\begin{tikzcd}
 & & b \\
b \times c \arrow[urr, "\mathrm{p}_1"] \arrow[drr, "\mathrm{p}_2"'] \arrow[rr, dashed, "\langle \mathrm{p}_1, \mathrm{p}_2 \rangle"] & & b \times c \arrow[u, "\mathrm{p}_1"'] \arrow[d, "\mathrm{p}_2"] \\
 & & c
\end{tikzcd}$$

$$\begin{tikzcd}
b \arrow[r, "\mathrm{p}_1 = \mathrm{id}_b"] & b
\end{tikzcd}$$

Existential doctrines

Definition

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ is ***existential*** when

- reindexings along projections $\mathrm{p}_1 \colon b \times c \longrightarrow b$ have left adjoints

$$\begin{tikzcd}
D(b \times c) \arrow[r, shift left=1.5ex, "\exists_{\hat{2}}"] \arrow[r, phantom, "\perp"] & D(b) \arrow[l, shift left=1.5ex, "D(\mathrm{p}_1)"]
\end{tikzcd}$$

- each left adjoint satisfies FR: $\exists_{\hat{2}}(\alpha(\mathrm{p}_1) \wedge \beta(\mathrm{p}_1, \mathrm{p}_2)) = \alpha(\mathrm{p}_1) \wedge \exists_{\hat{2}}(\beta(\mathrm{p}_1, \mathrm{p}_2))$
- $\big(\exists_{\hat{2}}(\beta)\big)(f) = \exists_{\hat{2}}\big(\beta(f \times \mathrm{id}_c)\big)$ for $f \colon b' \longrightarrow b$ in $\mathcal{B}$

$$\begin{tikzcd}
 & b' \arrow[rr, "f"] & & b \\
b' \times c \arrow[ur, "\mathrm{p}'_1"] \arrow[dr, "\mathrm{p}'_2"'] \arrow[rr, dashed, "\langle f\mathrm{p}'_1, \mathrm{p}'_2\rangle"] & & b \times c \arrow[ur, "\mathrm{p}_1"'] \arrow[dr, "\mathrm{p}_2"] & \\
 & c \arrow[rr, "\mathrm{id}_c"] & & c
\end{tikzcd}$$

Existential doctrines

Definition

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ is ***existential*** when

- reindexings along projections $\mathrm{p}_1\colon b \times c \longrightarrow b$ have left adjoints

$$\begin{tikzcd} D(b \times c) \arrow[r, shift left=2, "\exists_{\hat{2}}"] \arrow[r, phantom, "\perp"] & D(b) \arrow[l, shift left=2, "D(\mathrm{p}_1)"] \end{tikzcd}$$

- each left adjoint satisfies FR: $\exists_{\hat{2}}(\alpha(\mathrm{p}_1) \wedge \beta(\mathrm{p}_1, \mathrm{p}_2)) = \alpha(\mathrm{p}_1) \wedge \exists_{\hat{2}}(\beta(\mathrm{p}_1, \mathrm{p}_2))$
- $\big(\exists_{\hat{2}}(\beta)\big)(f) = \exists_{\hat{2}}\big(\beta(f\mathrm{p}'_1, \mathrm{p}'_2)\big)$ for $f\colon b' \longrightarrow b$ in $\mathcal{B}$

$$\begin{tikzcd} & b' \arrow[rr, "f"] & & b \\ b' \times c \arrow[ur, "\mathrm{p}'_1"] \arrow[dr, "\mathrm{p}'_2"'] \arrow[rr, dashed, "\langle f\mathrm{p}'_1, \mathrm{p}'_2 \rangle"] & & b \times c \arrow[ur, "\mathrm{p}_1"'] \arrow[dr, "\mathrm{p}_2"] & \\ & c \arrow[rr, "\mathrm{id}_c"] & & c \end{tikzcd}$$

Existential doctrines

Definition

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ is ***existential*** when

- reindexings along projections $\mathrm{p}_1 \colon b \times c \longrightarrow b$ have left adjoints

$$\begin{tikzcd} D(b \times c) \arrow[r, shift left=2, "\exists_{\hat{2}}"] \arrow[r, phantom, "\perp"] & D(b) \arrow[l, shift left=2, "D(\mathrm{p}_1)"] \end{tikzcd}$$

- each left adjoint satisfies FR: $\exists_{\hat{2}}(\alpha(\mathrm{p}_1) \wedge \beta(\mathrm{p}_1, \mathrm{p}_2)) = \alpha(\mathrm{p}_1) \wedge \exists_{\hat{2}}(\beta(\mathrm{p}_1, \mathrm{p}_2))$
- $\big(\exists_{\hat{2}}(\beta(\mathrm{p}_1, \mathrm{p}_2))\big)(f\mathrm{p}'_1) = \exists_{\hat{2}}\big(\beta(f\mathrm{p}'_1, \mathrm{p}'_2)\big)$ for $f \colon b' \longrightarrow b$ in $\mathcal{B}$

$$\begin{tikzcd} & b' \arrow[rr, "f"] & & b \\ b' \times c \arrow[ur, "\mathrm{p}'_1"] \arrow[dr, swap, "\mathrm{p}'_2"] \arrow[rr, dashed, "\langle f\mathrm{p}'_1, \mathrm{p}'_2 \rangle"] & & b \times c \arrow[ur, swap, "\mathrm{p}_1"] \arrow[dr, "\mathrm{p}_2"] & \\ & c \arrow[rr, "\mathrm{id}_c"] & & c \end{tikzcd}$$

Existential doctrines, examples

Examples

$\mathcal{Ctx}_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}_{\exists\wedge}^{\mathcal{T}}} \mathcal{Pos}$ for $\mathcal{T}$ in the $\exists\wedge$-fragment

$\mathcal{Set}^{\mathrm{op}} \xrightarrow{\mathbf{P}} \mathcal{Pos}$

$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathcal{M}} \mathcal{Pos}$ for $\mathcal{M}$ the monos of a stable factorization system in $\mathcal{C}$

In particular, $\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{Pos}$ for $\mathcal{C}$ regular

$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Vrn}} \mathcal{Pos}$ for $\mathcal{A}$ with finite products and weak equalisers

$\mathcal{Set}^{\mathrm{op}} \xrightarrow{H^{(-)}} \mathcal{Pos}$ for H a frame

Existential doctrines on $\mathcal{B}$ as posets in $[\mathcal{B}^{\mathrm{op}}, \mathcal{Set}]$

Proposition

An inf-semilattice D in $[\mathcal{B}^{\mathrm{op}}, \mathcal{Set}]$ is an existential doctrine if and only if D has sups indexed by representable objects

Proof

Write $\text{よ}\colon \mathcal{B} \longrightarrow [\mathcal{B}^{\mathrm{op}}, \mathcal{Set}]$ for the Yoneda embedding

The functor $D(- \times c)\colon \mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{Pos}$ is the power $D^{\text{よ}(c)}$

The adjointness condition determines $\exists^{p_1}\colon D^{\text{よ}(c)} \longrightarrow D$ as the sup operation $\square$

Elementary doctrines

Definition

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ is ***elementary*** when

- each reindexing along $d_{b,c} \overset{\mathrm{df}}{=} \langle \mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_2 \rangle \colon b \times c \longrightarrow b \times c \times c$ has a left adjoint

$$\begin{tikzcd} D(b \times c) \arrow[r, bend left, "\exists^{d_{b,c}}"] \arrow[r, phantom, "\perp"] & D(b \times c \times c) \arrow[l, bend left, "D(d_{b,c})"] \end{tikzcd}$$

- the left adjoint satisfies FR: $\exists^{d_{b,c}}\Big(D(d_{b,c})(\alpha) \wedge \beta\Big) = \alpha \wedge \exists^{d_{b,c}}(\beta)$
- $\left(\exists^{d_{b,c}}\right)_b : D(- \times c) \xrightarrow{\cdot} D(- \times c \times c)$

Examples

$\mathcal{S}et^{\mathrm{op}} \xrightarrow{\mathbf{P}} \mathcal{P}os$

$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathcal{M}} \mathcal{P}os$ for $\mathcal{M}$ the monos of a stable factorization system in $\mathcal{C}$

Elementary doctrines

Theorem

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ *is elementary if and only if for every* b *in* $\mathcal{B}$ *there is* δ_b *in* $D(b \times b)$ *such that*

(a) $D(\mathrm{p}_1)(\beta) \wedge \delta_b \leq D(\mathrm{p}_2)(\beta)$ *for* β *in* $D(b)$

(b) $\top_b \leq D(\langle \mathrm{id}_b, \mathrm{id}_b \rangle)(\delta_b)$

(c) $D(\langle \mathrm{p}_1, \mathrm{p}_3 \rangle)(\delta_b) \wedge D(\langle \mathrm{p}_2, \mathrm{p}_4 \rangle)(\delta_c) \leq \delta_{b \times c}$

Elementary doctrines

Theorem

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ *is elementary if and only if for every* b *in* $\mathcal{B}$ *there is* δ_b *in* $D(b \times b)$ *such that*

(a) $D(\mathrm{p}_1)(\beta) \wedge \delta_b \leq D(\mathrm{p}_2)(\beta)$ *for* β *in* $D(b)$ $\qquad \beta(\mathrm{p}_1) \wedge \delta_b(\mathrm{p}_1, \mathrm{p}_2) \leq \beta(\mathrm{p}_2)$

(b) $\top_b \leq D(\langle \mathrm{id}_b, \mathrm{id}_b \rangle)(\delta_b)$ $\qquad \top_b \leq \delta_b(\mathrm{p}_1, \mathrm{p}_1)$

(c) $D(\langle \mathrm{p}_1, \mathrm{p}_3 \rangle)(\delta_b) \wedge D(\langle \mathrm{p}_2, \mathrm{p}_4 \rangle)(\delta_c) \leq \delta_{b \times c}$ $\qquad \delta_b(\mathrm{p}_1, \mathrm{p}_3) \wedge \delta_c(\mathrm{p}_2, \mathrm{p}_4) \leq \delta_{b \times c}$

Elementary doctrines

Theorem

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ *is elementary if and only if for every* b *in* $\mathcal{B}$ *there is* δ_b *in* $D(b \times b)$ *such that*

(a) $D(\mathrm{p}_1)(\beta) \wedge \delta_b \leq D(\mathrm{p}_2)(\beta)$ *for* β *in* $D(b)$ $\qquad \beta(\mathrm{p}_1) \wedge \delta_b(\mathrm{p}_1, \mathrm{p}_2) \leq \beta(\mathrm{p}_2)$

(b) $\top_b \leq D(\langle \mathrm{id}_b, \mathrm{id}_b \rangle)(\delta_b)$ $\qquad \top_b \leq \delta_b(\mathrm{p}_1, \mathrm{p}_1)$

(c) $D(\langle \mathrm{p}_1, \mathrm{p}_3 \rangle)(\delta_b) \wedge D(\langle \mathrm{p}_2, \mathrm{p}_4 \rangle)(\delta_c) \leq \delta_{b \times c}$ $\qquad \delta_b(\mathrm{p}_1, \mathrm{p}_3) \wedge \delta_c(\mathrm{p}_2, \mathrm{p}_4) \leq \delta_{b \times c}$

$$\delta_b(\mathrm{p}_1, \mathrm{p}_3) \wedge \delta_c(\mathrm{p}_2, \mathrm{p}_4) \leq \delta_{b \times c}(\langle \mathrm{p}_1, \mathrm{p}_2 \rangle, \langle \mathrm{p}_3, \mathrm{p}_4 \rangle)$$

$$\begin{tikzcd}
 & & b \\
 & b \times c \arrow[ur] \arrow[dr] & \\
 & & c \\
b \times c \times b \times c \arrow[uurr, bend left, "\mathrm{p}_1"] \arrow[uur] \arrow[urr, "\mathrm{p}_2"'] \arrow[drr, "\mathrm{p}_3"] \arrow[ddr] \arrow[dddrr, bend right, "\mathrm{p}_4"'] & & \\
 & & b \\
 & b \times c \arrow[ur] \arrow[dr] & \\
 & & c
\end{tikzcd}$$

Elementary doctrines

Theorem

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ *is elementary if and only if for every* b *in* $\mathcal{B}$ *there is* δ_b *in* $D(b \times b)$ *such that*

(a) $D(\mathrm{p}_1)(\beta) \wedge \delta_b \leq D(\mathrm{p}_2)(\beta)$ *for* β *in* $D(b)$ $\qquad \beta(\mathrm{p}_1) \wedge \delta_b(\mathrm{p}_1, \mathrm{p}_2) \leq \beta(\mathrm{p}_2)$

(b) $\top_b \leq D(\langle \mathrm{id}_b, \mathrm{id}_b \rangle)(\delta_b)$ $\qquad \top_b \leq \delta_b(\mathrm{p}_1, \mathrm{p}_1)$

(c) $D(\langle \mathrm{p}_1, \mathrm{p}_3 \rangle)(\delta_b) \wedge D(\langle \mathrm{p}_2, \mathrm{p}_4 \rangle)(\delta_c) \leq \delta_{b \times c}$ $\qquad \delta_b(\mathrm{p}_1, \mathrm{p}_3) \wedge \delta_c(\mathrm{p}_2, \mathrm{p}_4) \leq \delta_{b \times c}$

$\delta_b(\mathrm{p}_1, \mathrm{p}_3) \wedge \delta_c(\mathrm{p}_2, \mathrm{p}_4) \leq \delta_{b \times c}(\langle \mathrm{p}_1, \mathrm{p}_2 \rangle, \langle \mathrm{p}_3, \mathrm{p}_4 \rangle)$

The left adjoint to $D(d_{b,c})\colon D(b \times c) \longrightarrow D(b \times c \times c)$ *is given by*

$$\begin{tikzcd}
D(b \times c) \arrow[r, "\exists^{d_{b,c}}"] & D(b \times c \times c) \\
\beta \arrow[r, mapsto] & D(\langle \mathrm{p}_1, \mathrm{p}_2 \rangle)(\beta) \wedge D(\langle \mathrm{p}_2, \mathrm{p}_3 \rangle)(\delta_b)
\end{tikzcd}$$

Elementary doctrines

Theorem

A primary doctrine $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ *is elementary if and only if for every* b *in* $\mathcal{B}$ *there is* δ_b *in* $D(b \times b)$ *such that*

(a) $D(\mathrm{p}_1)(\beta) \wedge \delta_b \leq D(\mathrm{p}_2)(\beta)$ *for* β *in* $D(b)$ $\qquad \beta(\mathrm{p}_1) \wedge \delta_b(\mathrm{p}_1, \mathrm{p}_2) \leq \beta(\mathrm{p}_2)$

(b) $\top_b \leq D(\langle \mathrm{id}_b, \mathrm{id}_b \rangle)(\delta_b)$ $\qquad \top_b \leq \delta_b(\mathrm{p}_1, \mathrm{p}_1)$

(c) $D(\langle \mathrm{p}_1, \mathrm{p}_3 \rangle)(\delta_b) \wedge D(\langle \mathrm{p}_2, \mathrm{p}_4 \rangle)(\delta_c) \leq \delta_{b \times c}$ $\qquad \delta_b(\mathrm{p}_1, \mathrm{p}_3) \wedge \delta_c(\mathrm{p}_2, \mathrm{p}_4) \leq \delta_{b \times c}$

$\delta_b(\mathrm{p}_1, \mathrm{p}_3) \wedge \delta_c(\mathrm{p}_2, \mathrm{p}_4) \leq \delta_{b \times c}(\langle \mathrm{p}_1, \mathrm{p}_2 \rangle, \langle \mathrm{p}_3, \mathrm{p}_4 \rangle)$

The left adjoint to $D(d_{b,c})\colon D(b \times c) \longrightarrow D(b \times c \times c)$ *is given by*

$$\begin{tikzcd}
D(b \times c) \arrow[r, "\exists^{d_{b,c}}"] & D(b \times c \times c) \\
\beta \arrow[r, mapsto] & D(\langle \mathrm{p}_1, \mathrm{p}_2 \rangle)(\beta) \wedge D(\langle \mathrm{p}_2, \mathrm{p}_3 \rangle)(\delta_b)
\end{tikzcd}$$

$$\beta(\mathrm{p}_1, \mathrm{p}_2) \wedge \delta_b(\mathrm{p}_2, \mathrm{p}_3)$$

Elementary doctrines, more examples

$\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}^{\mathcal{T}}_{\wedge=}} \mathcal{P}os$ for $\mathcal{T}$ in the $\wedge=$-fragment

$\delta_{(x_1:T_1,\dots,x_n:T_n)} \overset{\mathrm{df}}{=} \bigwedge_{i=1}^{n} (x_i = x_i')$

(a) $\beta(\mathrm{p}_1) \wedge \delta_b(\mathrm{p}_1, \mathrm{p}_2) \leq \beta(\mathrm{p}_2)$ for β in $D(b)$

(b) $\top_b \leq \delta_b(\mathrm{p}_1, \mathrm{p}_1)$

(c) $\delta_b(\mathrm{p}_1, \mathrm{p}_3) \wedge \delta_c(\mathrm{p}_2, \mathrm{p}_4) \leq \delta_{b\times c}(\langle \mathrm{p}_1, \mathrm{p}_2\rangle, \langle \mathrm{p}_3, \mathrm{p}_4\rangle)$

Elementary doctrines, more examples

$\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}_{\wedge=}^{\mathcal{T}}} \mathcal{P}os$ for $\mathcal{T}$ in the $\wedge=$-fragment

$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{P}os$ for $\mathcal{C}$ with finite limits

$\delta_c \overset{\mathrm{df}}{=} \langle \mathrm{id}_c, \mathrm{id}_c \rangle$

$$\begin{tikzcd}
a \arrow[rr, "\beta"] \arrow[dr, "{\langle \beta, \mathrm{id}_a \rangle}"] \arrow[dd, "{\langle \mathrm{id}_a, \beta \rangle}"'] & & b \arrow[dd, "{\langle \mathrm{id}_b, \mathrm{id}_b \rangle}"] \\
& b \times a \arrow[dr, "\mathrm{id}_b \times \beta"] & \\
a \times b \arrow[rr, "\beta \times \mathrm{id}_b"'] & & b \times b
\end{tikzcd}$$

(a) $D(\mathrm{p}_1)(\beta) \wedge \delta_b \leq D(\mathrm{p}_2)(\beta)$ for β in $D(b)$

(b) $\top_b \leq D(\langle \mathrm{id}_b, \mathrm{id}_b \rangle)(\delta_b)$

(c) $D(\langle \mathrm{p}_1, \mathrm{p}_3 \rangle)(\delta_b) \wedge D(\langle \mathrm{p}_2, \mathrm{p}_4 \rangle)(\delta_c) \leq \delta_{b \times c}$

Elementary doctrines, more examples

$\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}^{\mathcal{T}}_{\wedge=}} \mathcal{P}os$ for $\mathcal{T}$ in the $\wedge=$-fragment

$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{P}os$ for $\mathcal{C}$ with finite limits

$\delta_c \overset{\mathrm{df}}{=} \langle \mathrm{id}_c, \mathrm{id}_c \rangle$

$$\begin{tikzcd}
a \arrow[rr, "\beta"] \arrow[dd, "{\langle \mathrm{id}_a, \beta \rangle}"'] \arrow[dr, "{\langle \beta, \mathrm{id}_a \rangle}"] & & b \arrow[dd, "{\langle \mathrm{id}_b, \mathrm{id}_b \rangle}"] \\
& b \times a \arrow[dr, "\mathrm{id}_b \times \beta"] & \\
a \times b \arrow[rr, "\beta \times \mathrm{id}_b"'] & & b \times b
\end{tikzcd}$$

(a) $\mathrm{p}_1{}^{-1}(\beta) \wedge \delta_b \leq \mathrm{p}_2{}^{-1}(\beta)$ for β in $D(b)$

(b) $\top_b \leq \langle \mathrm{id}_b, \mathrm{id}_b \rangle^{-1}(\delta_b)$

(c) $\langle \mathrm{p}_1, \mathrm{p}_3 \rangle^{-1}(\delta_b) \wedge \langle \mathrm{p}_2, \mathrm{p}_4 \rangle^{-1}(\delta_c) \leq \delta_{b \times c}$

Elementary doctrines, more examples

$\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}^{\mathcal{T}}_{\wedge =}} \mathcal{P}os$ for $\mathcal{T}$ in the $\wedge =$-fragment

$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{P}os$ for $\mathcal{C}$ with finite limits

$\delta_c \overset{\mathrm{df}}{=} \langle \mathrm{id}_c, \mathrm{id}_c \rangle$

$$\begin{tikzcd}
b \arrow[r, "\mathrm{id}_b"] \arrow[d, "\mathrm{id}_b"'] & b \arrow[d, "\langle \mathrm{id}_b, \mathrm{id}_b \rangle"] \\
b \arrow[r, "\langle \mathrm{id}_b, \mathrm{id}_b \rangle"'] & b \times b
\end{tikzcd}$$

(a) $\mathrm{p}_1{}^{-1}(\beta) \wedge \delta_b \leq \mathrm{p}_2{}^{-1}(\beta)$ for β in $D(b)$

(b) $\top_b \leq \langle \mathrm{id}_b, \mathrm{id}_b \rangle^{-1}(\delta_b)$

(c) $\langle \mathrm{p}_1, \mathrm{p}_3 \rangle^{-1}(\delta_b) \wedge \langle \mathrm{p}_2, \mathrm{p}_4 \rangle^{-1}(\delta_c) \leq \delta_{b \times c}$

Elementary doctrines, more examples

$\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}^{\mathcal{T}}_{\wedge=}} \mathcal{P}os$ for $\mathcal{T}$ in the $\wedge$=-fragment

$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{P}os$ for $\mathcal{C}$ with finite limits

$\delta_c \overset{\mathrm{df}}{=} \langle \mathrm{id}_c, \mathrm{id}_c \rangle$

$$\begin{tikzcd}
b \times c \arrow[r, "{\langle \mathrm{p}_1, \mathrm{p}_1, \mathrm{p}_2 \rangle}"] \arrow[d, "{\langle \mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_2 \rangle}"'] & b \times b \times c \arrow[d, "{\langle \mathrm{p}_1, \mathrm{p}_3, \mathrm{p}_2, \mathrm{p}_3 \rangle}"] \\
b \times c \times c \arrow[r, "{\langle \mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_1, \mathrm{p}_3 \rangle}"'] & b \times c \times b \times c
\end{tikzcd}$$

(a) $\mathrm{p}_1{}^{-1}(\beta) \wedge \delta_b \leq \mathrm{p}_2{}^{-1}(\beta)$ for β in $D(b)$

(b) $\top_b \leq \langle \mathrm{id}_b, \mathrm{id}_b \rangle^{-1}(\delta_b)$

(c) $\langle \mathrm{p}_1, \mathrm{p}_3 \rangle^{-1}(\delta_b) \wedge \langle \mathrm{p}_2, \mathrm{p}_4 \rangle^{-1}(\delta_c) \leq \delta_{b \times c}$

Elementary doctrines, more examples

$\mathcal{Ctx}_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}_{\wedge=}^{\mathcal{T}}} \mathcal{Pos}$ for $\mathcal{T}$ in the $\wedge=$-fragment

$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{Sub}} \mathcal{Pos}$ for $\mathcal{C}$ with finite limits

$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Vrn}} \mathcal{Pos}$ for $\mathcal{A}$ with finite products and weak pullbacks

$\mathcal{Set}^{\mathrm{op}} \xrightarrow{H^{(-)}} \mathcal{Pos}$ for H an inf-semilattice with a least element

$$\delta_S \overset{\mathrm{df}}{=} \langle s_1, s_2 \rangle \mapsto \begin{cases} \top & \text{if } s_1 = s_2 \\ \bot & \text{otherwise} \end{cases}$$

Four 2-categories

$\mathit{Idx}\mathcal{P}\mathit{os}$

$$\begin{tikzcd}
\mathcal{B} \arrow[d, shift right=4, "\overline{f}"'] \arrow[d, shift left=4, "\overline{g}"] \arrow[d, phantom, "\xrightarrow[\theta]{\bullet}"] \\
\mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}^{\mathrm{op}} \arrow[dd, shift right=4, "\overline{f}^{\mathrm{op}}"'] \arrow[dd, shift left=4, "\overline{g}^{\mathrm{op}}"] \arrow[dd, phantom, "\xleftarrow[\theta^{\mathrm{op}}]{\bullet}"] \arrow[rrd, "D"] & & \\
 & \hat{f}\big\downarrow \cdot \leq \cdot \big\downarrow \hat{g} & \mathcal{P}os \\
\mathcal{C}^{\mathrm{op}} \arrow[rru, "E"'] & &
\end{tikzcd}$$

Four 2-categories

$$\begin{tikzcd}
\mathcal{I}dx\mathcal{P}os \\
\\
\mathcal{P}rim\mathcal{D}tn \arrow[uu]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B} \arrow[d, shift right=3, "\overline{f}"'] \arrow[d, shift left=3, "\overline{g}"] \arrow[d, phantom, "\overset{\bullet}{\underset{\overline{\theta}}{\rightarrow}}"] \\
\mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}^{\mathrm{op}} \arrow[d, shift right=3, "\overline{f}^{\mathrm{op}}"'] \arrow[d, shift left=3, "\overline{g}^{\mathrm{op}}"] \arrow[d, phantom, "\overset{\bullet}{\underset{\overline{\theta}^{\mathrm{op}}}{\leftarrow}}"] \arrow[rr, "D"] & \hat{f}\downarrow \cdot \leq \cdot \downarrow \hat{g} & \mathcal{P}os \\
\mathcal{C}^{\mathrm{op}} \arrow[urr, "E"']
\end{tikzcd}$$

0-cells: D primary doctrine

1-cells: $\mathcal{B} \xrightarrow{\overline{f}} \mathcal{C}$ preserves finite products

each component of $\hat{f}\colon D \overset{\bullet}{\longrightarrow} E\overline{f}^{\mathrm{op}}$ preserves finite meets

Four 2-categories

$$\begin{tikzcd} \mathcal{IdxPos} \\ \\ \mathcal{PrimDtn} \arrow[uu] \end{tikzcd}
\qquad
\begin{tikzcd} \mathcal{B} \arrow[d, shift right=4, "\overline{f}"'] \arrow[d, shift left=4, "\overline{g}"] \arrow[d, phantom, "\underset{\theta}{\overset{\bullet}{\rightarrow}}"] \\ \mathcal{C} \end{tikzcd}
\qquad
\begin{tikzcd} \mathcal{B}^{op} \arrow[dd, shift right=4, "\overline{f}^{op}"'] \arrow[dd, shift left=4, "\overline{g}^{op}"] \arrow[dd, phantom, "\underset{\theta^{op}}{\overset{\bullet}{\leftarrow}}"] \arrow[drr, "D"] & & \\ & \hat{f} \Big\downarrow \cdot \le \cdot \Big\downarrow \hat{g} & \mathcal{Pos} \\ \mathcal{C}^{op} \arrow[urr, "E"'] & & \end{tikzcd}$$

0-cells: D primary doctrine

1-cells: $\mathcal{B} \xrightarrow{\overline{f}} \mathcal{C}$ preserves finite products

each component of $\hat{f} : D \xrightarrow{\bullet} E\overline{f}^{op}$ preserves finite meets

$$\begin{tikzcd} D(b) \times D(b) \arrow[r, "\wedge_b"'] \arrow[d, "\hat{f}_b \times \hat{f}_b"'] & D(b) \arrow[d, "\hat{f}_b"'] & 1 \arrow[l, "\top"'] \arrow[dl, "\top"] \\ E(\overline{f}(b)) \times E(\overline{f}(b)) \arrow[r, "\wedge_{\overline{f}(b)}"] & E(\overline{f}(b)) & \end{tikzcd}$$

Four 2-categories

$$\begin{tikzcd} \mathcal{IdxPos} \\ \\ \mathcal{PrimDtn} \arrow[uu] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{B} \arrow[d, shift right=3, "\overline{f}"'] \arrow[d, shift left=3, "\overline{g}"] \arrow[d, phantom, "\overset{\cdot}{\underset{\theta}{\rightarrow}}"] \\ \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{B}^{\mathrm{op}} \arrow[dd, shift right=3, "\overline{f}^{\mathrm{op}}"'] \arrow[dd, shift left=3, "\overline{g}^{\mathrm{op}}"] \arrow[dd, phantom, "\overset{\cdot}{\underset{\theta^{\mathrm{op}}}{\leftarrow}}"] \arrow[rrd, "D"] & & \\ & \hat{f}\Big\downarrow\cdot \leq \cdot\Big\downarrow\hat{g} & \mathcal{Pos} \\ \mathcal{C}^{\mathrm{op}} \arrow[rru, "E"'] & & \end{tikzcd}$$

0-cells: D primary doctrine

1-cells: $\mathcal{B} \xrightarrow{\overline{f}} \mathcal{C}$ preserves finite products

each component of $\hat{f}\colon D \xrightarrow{\cdot} E\overline{f}^{\mathrm{op}}$ preserves finite meets

2-cells: those in $\mathcal{IdxPos}$

Four 2-categories

$$\begin{tikzcd}
\mathcal{IdxPos} \\
\mathcal{PrimDtn} \arrow[u] \\
\mathcal{ElemDtn} \arrow[u, hook]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B} \arrow[d, shift right=3, "\overline{f}"'] \arrow[d, shift left=3, "\overline{g}"] \arrow[d, phantom, "\underset{\theta}{\overset{\cdot}{\rightarrow}}"] \\
\mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B}^{\mathrm{op}} \arrow[dd, shift right=3, "\overline{f}^{\mathrm{op}}"'] \arrow[dd, shift left=3, "\overline{g}^{\mathrm{op}}"] \arrow[dd, phantom, "\underset{\theta^{\mathrm{op}}}{\overset{\cdot}{\leftarrow}}"] \arrow[drr, "D"] & & \\
& \hat{f}\big\downarrow \cdot \leq \cdot \big\downarrow \hat{g} & \mathcal{Pos} \\
\mathcal{C}^{\mathrm{op}} \arrow[urr, "E"'] & &
\end{tikzcd}$$

0-cells: D elementary doctrine

1-cells: those in $\mathcal{PrimDtn}$ whose components preserve $\exists_d$'s (eqv. δ's)

2-cells: those in $\mathcal{IdxPos}$

Four 2-categories

$$\begin{tikzcd}
\mathcal{IdxPos} \\
\mathcal{PrimDtn} \arrow[u] \\
\mathcal{ElemDtn} \arrow[u, hook] & & \mathcal{ExistDtn} \arrow[llu, hook]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B} \arrow[d, shift right=3, "\overline{f}"'] \arrow[d, shift left=3, "\overline{g}"] \arrow[d, phantom, "\overset{\bullet}{\underset{\theta}{\rightarrow}}"] \\
\mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B}^{\mathrm{op}} \arrow[d, shift right=3, "\overline{f}^{\mathrm{op}}"'] \arrow[d, shift left=3, "\overline{g}^{\mathrm{op}}"] \arrow[d, phantom, "\overset{\bullet}{\underset{\theta^{\mathrm{op}}}{\leftarrow}}"] \arrow[r, "D"] \arrow[r, phantom, shift right=6, "\hat{f}\downarrow\cdot\leq\cdot\downarrow\hat{g}"'] & \mathcal{Pos} \\
\mathcal{C}^{\mathrm{op}} \arrow[ru, "E"']
\end{tikzcd}$$

0-cells: D existential doctrine
1-cells: those in $\mathcal{PrimDtn}$ whose components preserve $\exists$’s
2-cells: those in $\mathcal{IdxPos}$

Four 2-categories

$$\begin{tikzcd}
\mathcal{IdxPos} & & \\
\mathcal{PrimDtn} \arrow[u] & & \\
\mathcal{ElemDtn} \arrow[u, hook] & & \mathcal{ExistDtn} \arrow[llu, hook] \\
& \mathcal{ExistElemDtn} \arrow[lu, hook] \arrow[ru, hook] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B} \arrow[d, shift right=3ex, "\overline{f}"'] \arrow[d, shift left=3ex, "\overline{g}"] \arrow[d, phantom, "\xrightarrow[\overline{\theta}]{\cdot}"] \\
\mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B}^{\mathrm{op}} \arrow[dd, shift right=3ex, "\overline{f}^{\mathrm{op}}"'] \arrow[dd, shift left=3ex, "\overline{g}^{\mathrm{op}}"] \arrow[dd, phantom, "\xleftarrow[\overline{\theta}^{\mathrm{op}}]{\cdot}"] \arrow[drr, "D"] & & \\
& \hat{f}\Big\downarrow \cdot \leq \cdot \Big\downarrow \hat{g} & \mathcal{P}os \\
\mathcal{C}^{\mathrm{op}} \arrow[urr, "E"'] & &
\end{tikzcd}$$

“Doctrines” for doctrines

Theorem

(i) $\mathcal{PrimDtn} \longrightarrow \mathcal{IdxPos}$ *are the pseudoalgebras for a lax idempotent 2-monad on* $\mathcal{IdxPos}$

(ii) $\mathcal{ExistDtn} \longrightarrow \mathcal{PrimDtn}$ *are the pseudoalgebras for a lax idempotent 2-monad on* $\mathcal{PrimDtn}$

(iii) $\mathcal{ElemDtn} \longrightarrow \mathcal{PrimDtn}$ *are the pseudoalgebras for a colax idempotent 2-comonad on* $\mathcal{PrimDtn}$

(iv) $\mathcal{ExistDtn} \longrightarrow \mathcal{IdxPos}$ *are the pseudoalgebras for a lax idempotent 2-monad on* $\mathcal{IdxPos}$

F. Pasquali
A co-free construction for elementary doctrines *Appl.Categ. Structures* 2015
J. Emmenegger, F. Pasquali, G. R.
Elementary doctrines as coalgebras *J.Pure Appl.Algebra* 2020
D. Trotta
The existential completion *Theory Appl. Categ.* 2020

The endofunctor of the comonad of elementary doctrines

Given $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}rd$ a primary doctrine

the primary doctrine $\mathrm{Des}_D \colon \mathcal{E}_D^{\mathrm{op}} \longrightarrow \mathcal{P}os$ is

Objects of $\mathcal{E}_D$ are (b, ρ)

- b is an object of $\mathcal{B}$
- ρ is an object of $D(b \times b)$ such that
  - $\top_A \leq D(\langle \mathrm{id}_A, \mathrm{id}_A \rangle)(\rho)$
  - $\rho \leq D(\langle \mathrm{p}_2, \mathrm{p}_1 \rangle)(\rho)$
  - $D(\langle \mathrm{p}_1, \mathrm{p}_2 \rangle)(\rho) \wedge D(\langle \mathrm{p}_2, \mathrm{p}_3 \rangle)(\rho) \leq D(\langle \mathrm{p}_1, \mathrm{p}_3 \rangle)(\rho)$

Arrows of $\mathcal{E}_D$ are $f \colon (b, \rho) \longrightarrow (b', \rho')$

- $f \colon b \longrightarrow b'$ in $\mathcal{B}$
- $\rho \leq D(f \times f)(\rho')$

$$\mathrm{Des}_D(b, \rho) \overset{\mathrm{df}}{=} \{\beta \in D(b) \mid D(\mathrm{p}_1)(\beta) \wedge \rho \leq D(\mathrm{p}_2)(\beta)\} \subseteq D(b)$$

The endofunctor of the comonad of elementary doctrines

Given $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}rd$ a primary doctrine

the primary doctrine $\mathrm{Des}_D\colon \mathcal{E}_D^{\mathrm{op}} \longrightarrow \mathcal{P}os$ is

Objects of $\mathcal{E}_D$ are (b, ρ)

- b is an object of $\mathcal{B}$
- ρ is an object of $D(b \times b)$ such that
  - $\top_A \leq \rho(\mathrm{p}_1, \mathrm{p}_1)$
  - $\rho(\mathrm{p}_1, \mathrm{p}_2) \leq \rho(\mathrm{p}_2, \mathrm{p}_1)$
  - $\rho(\mathrm{p}_1, \mathrm{p}_2) \wedge \rho(\mathrm{p}_2, \mathrm{p}_3) \leq \rho(\mathrm{p}_1, \mathrm{p}_3)$

Arrows of $\mathcal{E}_D$ are $f\colon (b, \rho) \longrightarrow (b', \rho')$

- $f\colon b \longrightarrow b'$ in $\mathcal{B}$
- $\rho(\mathrm{p}_1, \mathrm{p}_2) \leq \rho'(f\mathrm{p}_1, f\mathrm{p}_2)$

$$\mathrm{Des}_D(b, \rho) \overset{\mathrm{df}}{=} \{\beta \in D(b) \mid \beta(\mathrm{p}_1) \wedge \rho(\mathrm{p}_1, \mathrm{p}_2) \leq \beta(\mathrm{p}_2)\} \subseteq D(b)$$

On the notion of set

“I say that a manifold (a collection, a set) of elements that belong to any conceptual sphere is *well-defined*, when on the basis of its definition and as a consequence of the logical principle of excluded middle, it must be regarded as internally determined *both* whether an object pertaining to the same conceptual sphere belongs or not as an element to the manifold, *and* whether two objects belonging to the set are equal to each other or not, despite formal differences in the ways they are given”

G. Cantor

Über unendliche lineare Punktmannigfaltigkeiten *Math. Ann.* 1879

“A set is not an entity which has an ideal existence: a set exists only when it has been defined. To define a set we prescribe, at least implicitly, what we (the constructing intelligence) must do in order to construct an element of the set, and what we must do to show that two elements are equal”

E. Bishop

Foundations of Constructive Analysis 1967

Instances of $\mathrm{Des}_D\colon \mathcal{E}_D^{\mathrm{op}} \longrightarrow \mathcal{P}os$

$$\mathcal{C}tx_{\mathcal{L}}^{\mathrm{op}} \xrightarrow{\mathrm{LT}^{\mathcal{T}}} \mathcal{P}os \qquad \mathcal{C}tx_{\mathcal{L}'}^{\mathrm{op}} \xrightarrow{\mathrm{LT}^{\mathcal{T}^{\mathrm{eq}}}} \mathcal{P}os$$

$$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Vrn}} \mathcal{P}os$$

$$\begin{tikzcd}
\mathcal{E}_{\mathrm{Vrn}}^{\mathrm{op}} \arrow[r, two heads] \arrow[rr, two heads, "\mathrm{Des}_{\mathrm{Vrn}}"'] & (\mathcal{A}_{\mathrm{ex}})^{\mathrm{op}} \arrow[r, two heads, "\mathrm{Sub}"] & \mathcal{P}os
\end{tikzcd}$$

$$\mathrm{FuzSet}(H)^{\mathrm{op}} \xrightarrow{\int(H^{(-)})} \mathcal{P}os$$

$$\begin{tikzcd}
\mathcal{E}_{\int(H^{(-)})}^{\mathrm{op}} \arrow[r, two heads] \arrow[rr, two heads, "\mathrm{Des}_{\int(H^{(-)})}"'] & \mathrm{sep}_{\mathrm{can}}([H^{\mathrm{op}}, \mathcal{S}et])^{\mathrm{op}} \arrow[r, two heads, "\mathrm{ClSub}"] & \mathcal{P}os
\end{tikzcd}$$

B. Poizat
Une théorie de Galois imaginaire *J. Symb. Log.* 1983
A. Carboni, E. Vitale
Regular and exact completions *J.Pure Appl.Algebra* 1998
M.P. Fourman, D.S. Scott
Sheaves and logic *Applications of sheaves* 1979
G.P. Monro
Quasitopoi, logic and Heyting valued-models *J.Pure Appl.Algebra* 1986
O. Wyler
Lecture notes on topoi and quasitopoi 1991

Explaining the quotient

The quotients $\mathcal{E}_{\mathrm{Vrn}} \twoheadrightarrow \mathcal{A}_{\mathrm{ex}}$ and $\mathcal{E}_{\int(H^{(-)})} \twoheadrightarrow \mathrm{sep}([H^{\mathrm{op}}, \mathcal{S}et])$ identify arrows exactly when they have the same actions on "parametrized propositions"

Proposition

Let $\mathcal{B}^{\mathrm{op}} \xrightarrow{D} \mathcal{P}os$ *be an elementary doctrine.*
For $f, g\colon b \longrightarrow c$ *the following are equivalent*

(i) $\beta(f\mathrm{p}_1, \mathrm{p}_2) = \beta(g\mathrm{p}_1, \mathrm{p}_2)$ *for every* b' *in* $\mathcal{B}$ *and* β *in* $D(c \times b')$

(ii) $\top_b \leq \delta_c(f\mathrm{p}_1, g\mathrm{p}_1)$

M.E. Maietti, G. R.
Unifying Exact Completions *Appl.Categ. Structures* 2013
C.J. Cioffo
Homotopy setoids and generalized quotient completion *PhD thesis*, 2022

Università
di Genova