Workshop on Doctrines & Fibrations
Departimento di Matematica "Tullio Levi-Civita"
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Strictifying Path Categories

Joint work in progress with
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Speaker
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Locally cartesian closed categories

A category $\mathcal{C}$ is locally cartesian closed if it has finite limits and if every re-indexing functor:

$$\mathcal{C}/\Gamma \xrightarrow{(-)[f]} \mathcal{C}/\Delta$$

along some arrow $\Delta \xrightarrow{f} \Gamma$ has a right adjoint.

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- Sections of $\Gamma.A \to \Gamma$ are semantic terms of A in context Γ.

Seely, *Locally cartesian closed categories and type theory*, 1983.

Such a category $\mathcal{C}$ is a *non-genuine* model of a dependent type theory with extensional $=$, Π, Σ.

Σ types in $\mathcal{C}$

- *Formation.* If:

$$\Gamma.A \to \Gamma \text{ and } \Gamma.A.B \to \Gamma.A$$

represent the type judgements:

$$\Gamma \vdash A : \textsc{Type} \text{ and } \Gamma, x : A \vdash B : \textsc{Type}$$

then:

$$(\ \Gamma.\mathbf{\Sigma}AB \to \Gamma\ ) := (\ \Gamma.A.B \to \Gamma.A \to \Gamma\ )$$

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- *Elimination and Computation.* In some way is done.

Substitution? Issues!

If we are given $\Delta \xrightarrow{f} \Gamma$ and $\Gamma.A \to \Gamma$ then the judgement:

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is represented by the re-indexing $\Delta.A[f] \to \Delta$ of $\Gamma.A \to \Gamma$ along f.

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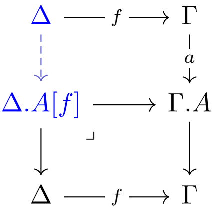

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But then, if we are given $\Omega \xrightarrow{g} \Delta \xrightarrow{f} \Gamma$:

- $\Omega.A[f][g] \cong \Omega.A[fg]$ and not necessarily $\Omega.A[f][g] \equiv \Omega.A[fg]$

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- $\Omega.A[f][g] \cong \Omega.A[fg]$ and not necessarily $\Omega.A[f][g] \equiv \Omega.A[fg]$
- $$\begin{array}{ccc} & & \Omega.A[f][g] \\ & \nearrow^{a[f][g]} & \\ \Omega & & \wr\Vert \\ & \searrow_{a[fg]} & \\ & & \Omega.A[fg] \end{array}$$ and not necessarily $a[f][g] \equiv a[fg]$

In this sense $\mathcal{C}$ is not a genuine model.

Hofmann's coherence result

However, in:

Hofmann, *On the Interpretation of Type Theory in Locally Cartesian Closed Categories*, 1994.

every locally cartesian closed category is shown to be equivalent to a split comprehension category (still endowed with extensional $=$, Π and Σ).

Comprehension categories

If $\mathcal{C}$ is a category with a terminal object, then a comprehension category (p, χ) over $\mathcal{C}$ is a Grothendieck fibration $\mathcal{E} \xrightarrow{p} \mathcal{C}$ together with a fully faithful functor $\mathcal{E} \xrightarrow{\chi} \mathcal{C}^{\rightarrow}$ mapping cartesian morphisms to pullback squares and such that the diagram:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad \chi \quad} & \mathcal{C}^{\rightarrow} \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle \mathrm{cod}} & \\ & \mathcal{C} & \end{array}$$

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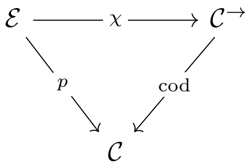

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If (p, χ) is equipped with the "right structure", then one can interpret in the non-genuine way the judgements of a dtt with extensional $=$, Π and Σ formally as for lccc's.

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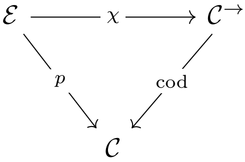

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If (p, χ) is equipped with the "right structure", then one can interpret in the non-genuine way the judgements of a dtt with extensional $=$, Π and Σ formally as for lccc's.

We say that (p, χ) is split if p is split. In this case (p, χ) is a genuine model, since $A[fg] \equiv A[f][g]$ "abstractly" in $\mathcal{E}$.

In general: *the right-adjoint coherence*

The inclusion:

$$\{\text{split cc over } \mathcal{C}\} \hookrightarrow \{\text{cc over } \mathcal{C}\}$$

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- maps every cc (p, χ) into a split cc equivalent to (p, χ);
- (under some pseudo-stability condition) preserves the semantic extensional $=$, Π and Σ structure.

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Hofmann's result is an instance of this one.

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The inclusion:

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also has left adjoint. It:

- maps every cc (p, χ) into a split cc equivalent to (p, χ) as well;
- (under some weak-stability condition) preserves the semantic intensional $=$, Π and Σ structure.

Lumsdaine, Warren, *The local universes model*, 2015.

Intensional identity types

Formation & Introduction rules.

$$\frac{\vdash A : \textsc{Type}}{\begin{array}{c} x, x' : A \vdash x = x' : \textsc{Type} \\ x : A \vdash r(x) : x = x \end{array}}$$

Path Elimination & Computation rules.

$$\frac{\begin{array}{c} \vdash A : \textsc{Type} \\ x, x' : A;\ p : x = x' \vdash C(x, x', p) : \textsc{Type} \\ x : A \vdash q(x) : C(x, x, r(x)) \end{array}}{\begin{array}{ll} x, x' : A;\ p : x = x' \vdash & J(q, x, x', p) : C(x, x', p) \\ x : A \vdash & J(q, x, x, r(x)) \equiv q(x) \end{array}}$$

Propositional identity types

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Propositional identity types in the literature

Propositional identity types appear in:

- Coquand, Danielsson, *Isomorphism is equality*, 2013.
- Bezem, Coquand, Huber, *A model of type theory in cubical sets*, 2014.
- van den Berg, *Path categories and propositional identity types*, 2018.

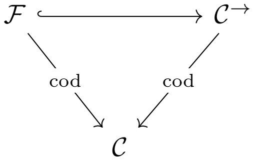
Path categories i.e. non-genuine models of propositional identity types

A path category $\mathcal{C}$ is a category with a terminal object, a class of fibrations and a class of weak equivalences such that the following properties are satisfied:

1. The composition of two fibrations is a fibration as well.
2. Every pullback of a fibration exists and is a fibration as well.
3. Every pullback of an acyclic fibration is a trivial fibration as well.
4. Weak equivalences satisfy 2-out-of-six.
5. Every isomorphism is a trivial fibration and every trivial fibration has a section.
6. For every object X of $\mathcal{C}$ there is an object PX, called path object on X, together with a weak equivalence $X \xrightarrow{r} PX$ and a fibration $PX \xrightarrow{\langle s,t \rangle} X \times X$ such that $(X \xrightarrow{r} PX \xrightarrow{\langle s,t \rangle} X \times X) = \delta_X$.
7. Every arrow of target a terminal object is a fibration.

Path categories as comprehension categories

Let $\mathcal{C}$ be a category with terminal object. Then a path categorical structure over $\mathcal{C}$ can be re-phrased as a comprehension category:



over $\mathcal{C}$ with:

- propositional = types
- strong Σ types
- contextuality/democracy

and vice versa. These mappings preserve equivalences (they seem to constitute a biequivalence).

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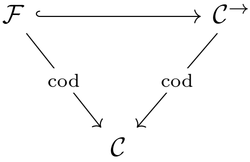

over $\mathcal{C}$ with:

- weakly stable propositional = types
- weakly stable strong Σ types
- contextuality/democracy

and vice versa. These mappings preserve equivalences (they seem to constitute a biequivalence).

Preservation of propositional $=$

In:

Bocquet, *Strictification of weakly stable type-theoretic structures using generic contexts*, 2021.

the 2-left adjoint splitting is proven to preserve the semantic propositional $=$ structure under the usual weak-stability conditions,

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Coherence for path categories

Theorem

Every path category admits a split replacement.