Algebraic structures for modified realizability (jww Marcus Briët)

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Section 1

Implicative algebras

Implicative structure

Implicative structures

An *implicative structure* is a complete poset $(A, \preccurlyeq)$ together with a binary operation $\to: A \times A \to A$ satisfying the following two conditions:

(1) If $a' \preccurlyeq a$ and $b \preccurlyeq b'$ then $a \to b \preccurlyeq a' \to b'$.

(2) For all subsets $B \subseteq A$ we have

$$a \to \bigcurlywedge_{b \in B} b = \bigcurlywedge_{b \in B} (a \to b).$$

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Intuition

We think of the elements of A as truth values or bits of evidence. However, we should *not* think of $\preccurlyeq$ as giving us the logical ordering of these truth values: it is more of an "evidential ordering" ("subtyping ordering").

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(1) If $a \in S$ and $a \preccurlyeq b$, then $b \in S$.

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(3) Both k and s belong to S.

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(3) Both k and s belong to S.

Here k and s are defined as follows:

$$\mathrm{k} \;:=\; \bigcurlywedge_{a,b} a \to b \to a$$

$$\mathrm{s} \;:=\; \bigcurlywedge_{a,b,c} (a \to b \to c) \to (a \to b) \to (a \to c)$$

Implicative algebras

Implicative algebra (Miquel)

A quadruple $(A, \preccurlyeq, \to, S)$ consisting of an implicative structure $(A, \preccurlyeq, \to)$ together with a separator S is called an *implicative algebra*.

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(1) A complete Heyting algebra with a distinguished filter.

(2) If P is a total combinatory algebra, then $\mathrm{Pow}(P)$ ordered by inclusion together with the implication

$$X \to Y = \{z \in P : (\forall x \in X)\, zx \in Y\}.$$

The inhabited subsets form a separator.

Tripos

Let us write PreHey for the category of *preHeyting algebras* (preorders whose poset reflections are Heyting algebras).

Tripos (Hyland, Johnstone, Pitts)

A tripos is a functor $P : \mathit{Sets} \to \text{PreHey}$ such that:

- for each function $f : Y \to X$, the operation $Pf : PX \to PY$ has both adjoints satisfying the Beck-Chevally condition.
- There is a set Σ and an element $\top \in P(\Sigma)$ such that for any $A \in P(X)$ there is some map $a : X \to \Sigma$ (not necessarily unique) such that $P(a)(\top) \cong A$.

The tripos-to-topos construction allows us to construct a topos from a tripos.

Implicative tripos

Proposition

Let $A = (A, \preccurlyeq, \to, S)$ be an implicative algebra. If we preorder A as follows:

$$a \vdash b :\Longleftrightarrow a \to b \in S,$$

then A carries the structure of a preHeyting algebra.

In this preHeyting algebra the implication is given by $\to$. We think of $\vdash$ as giving us the *logical ordering*.

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Indeed, if $A = (A, \preccurlyeq, \to, S)$ be an implicative algebra and X is a set, then we can consider A^X as an implicative algebra as well: implication and the order can be defined pointwise, while

$$\varphi : X \to A \in S^X :\Longleftrightarrow \bigcurlywedge_{x \in X} \varphi(x) \in S.$$

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Indeed, if we put $PX = (A^X, \vdash_{S^X})$, then this defines a tripos (the "implicative tripos").

Section 2

Arrow algebras

Arrow structure

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(3) S contains the combinators k, s and a.

Here a is the combinator:

$$\mathsf{a} \quad := \bigcurlywedge_{x, I, (y_i)_{i \in I}, (z_i)_{i \in I}} \Big(\bigcurlywedge_{i \in I} x \to y_i \to z_i\Big) \to x \to \Big(\bigcurlywedge_{i \in I} y_i \to z_i\Big).$$

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Indeed, if $A = (A, \preccurlyeq, \to, S)$ be an implicative algebra and X is a set, then we can consider A^X as an arrow algebra as well: implication and the order can be defined pointwise, with $\varphi : X \to A \in S^X$ if $\curlywedge_{x \in X} \varphi(x) \in S$. Indeed, if we put $PX = (A^X, \vdash_{S^X})$, then this defines a tripos (the "arrow tripos").

Section 3

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Partial applicative poset

A *partial applicative poset* (abbreviated *PAP*) is a triple $(A, \cdot, \leq)$ where $(A, \leq)$ is a poset and $\cdot$ is a partial binary operation which satisfies:

> if $a'b'$ is defined and $a \leq a'$ and $b \leq b'$, then ab is defined and $ab \leq a'b'$.

We say that A is *total* is the application operation is total, and A is *discrete* if the order $\leq$ is a discrete order.

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Examples

1. K_1: the set of natural numbers with Kleene application ($n \cdot m$ is the outcome of the n-th Turing machine on input m, whenever defined) and the discrete order.
2. Terms in the untyped λ-calculus and $M \leq N$ if $M \twoheadrightarrow_\beta N$.

Pcas

Filter

Let A be a partial applicative poset. A *filter* F on A is a subset $F \subseteq A$ such that:

(i) if $a, b \in F$ and ab is defined, then $ab \in F$.

(ii) if $a \leq b$ and $a \in F$, then $b \in F$.

(iii) there are elements $\mathsf{k}, \mathsf{s} \in F$ satisfying:

  (1) $\mathsf{k}ab = a$;
  (2) $\mathsf{s}ab\downarrow$;
  (3) if $ac(bc)\downarrow$, then $\mathsf{s}abc\downarrow$ and $\mathsf{s}abc \leq ac(bc)$,

  for all $a, b, c \in A$.

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Pcas

A *partial combinatory algebra* (abbreviated PCA) is a pair consisting of a partial applicative poset $(A, \cdot, \leq)$ and a filter $A^{\#}$ on it. The PCA will be called *total* or *discrete* if the underlying partial applicative poset is. The PCA will be called *absolute* if $A^{\#} = A$.

Arrow algebras from a pca

Proposition

If P is a pca, then the collection DP of downsets in P carries an arrow structure with

$$X \to Y := \{z \in P : (\forall x \in X)\, xz\downarrow \text{ and } xz \in Y\}.$$

In addition, $S = \{X \in DP : (\exists x \in X)\, x \in F\}$ is a separator on this arrow structure.

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Proposition

If P is a pca, then the collection $\mathrm{PER}(P)$ of subsets of $P \times P$ which are downwards closed, symmetric and transitive carries an arrow structure with

$$X \to Y := \{(z, z') \in P^2 : (\forall (x, x') \in X)\, xz\downarrow, x'z'\downarrow \text{ and } (xz, x'z') \in Y\}.$$

In addition, $S = \{X \in \mathrm{PER}(P) : (\exists (x, x') \in X)\, x, x' \in F\}$ is a separator on this arrow structure.

Section 4

Nuclei

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Nucleus

Let $A = (A, \preccurlyeq, \to, S)$ be an arrow algebra. A mapping $j : A \to A$ will be called a *nucleus* if the following three properties are satisfied:

(1) $a \preccurlyeq b$ implies $ja \preccurlyeq jb$ for all $a, b \in A$.

(2) $\curlywedge_{a \in A} a \to ja \in S$.

(3) $\curlywedge_{a,b \in A} (a \to jb) \to (ja \to jb) \in S$.

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Proposition

Let $(A, \preccurlyeq, \to, S)$ be an arrow algebra and $j : A \to A$ be a nucleus on it. Then $\mathcal{A}_j = (A, \preccurlyeq, \to_j, S_j)$ with

$$a \to_j b \quad :\equiv \quad a \to jb$$
$$a \in S_j \quad :\Leftrightarrow \quad ja \in S$$

is also an arrow algebra.

Section 5

Modified realizability

Modified realizability

Modified realizability is characterised by the following ideas:

- There is a distinction between actual and potential realizers.
- Every actual realizer is also a potential realizer, but not conversely.
- Every proposition, including $\perp$, has at least one potential realizer.
- Something is true if it has an actual realizer.

The first to define a modified realizability topos was Grayson, based on ideas by Hyland.

The modification of an arrow algebra

Let $A = (A, \preccurlyeq, \to, S)$ be an arrow algebra. Then we can define a new arrow algebra $A^{\to}$ as follows: its elements are pairs $x = (x_a, x_p) \in A^2$ with $x_a \preccurlyeq x_p$ (here p stands for potential and a for actual).

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$A^{\to}$ is *often* an arrow algebra:

1. we order the pairs pointwise.
2. implication is defined follows:

$$x \to y = (x_a \to y_a \curlywedge x_p \to y_p, x_p \to y_p).$$

3. an element $x \in A^{\to}$ belongs to the separator if x_a does.

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3. an element $x \in A^{\to}$ belongs to the separator if x_a does.

On $A^{\to}$ we can define a nucleus as follows:

$$j(x) = (x_a, x_p) \vee (\bot, \top).$$

(Here $\vee$ refers to the logical ordering.) The resulting arrow algebra can be called the *modification* of A.

Toposes for modified realizability

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In the internal logic of this topos for "extensional modified realizability" the characteristic principles of modified realizability hold:

$$\begin{array}{rcl} \mathsf{AC} & : & \forall x^\sigma \, \exists y^\tau \, \alpha(x,y) \to \exists f^{\sigma\to\tau} \, \forall x^\sigma \, \alpha(x, f(x)) \\ \mathsf{IP} & : & (\varphi \to \exists x^\sigma \, \psi) \to \exists x^\sigma (\varphi \to \psi) \end{array}$$

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This topos has various subtoposes in which these principles also hold. One such is studied in the MSc thesis of Mees de Vries.

THANK YOU!

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