Eliminating Imaginaries and adding structure: How does the geometric completion fit in?

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Completions of doctrines

For this presentation, a *doctrine* is any functor

$$P \colon \mathcal{C}^{op} \to \mathbf{PreOrd}.$$

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(i) Trotta's existential completion,

(ii) Coumans' canonical extension.

Other completions change the indexing category $\mathcal{C}$, e.g.

(iii) Cioffo's strictification of a biased doctrine,

(iv) the quotient completion of Maietti and Rosolini.

Overview

(a) First we will review the geometric completion of a doctrine,

(b) We then describe an infinitary exact completion for a geometric doctrine,

(c) Finally, we develop an abstract framework for completions of doctrines based on relative topos theory.

Geometric doctrines

Definition

A *geometric doctrine* over a **cartesian** category $\mathcal{C}$ is a functor

$$\mathbb{L} \colon \mathcal{C}^{op} \to \mathbf{Frm}$$

such that for each arrow f, $\mathbb{L}(f)$ has a left adjoint $\exists_f$ satisfying

(i) Frobenius reciprocity (i.e. $\mathbb{L}(f)$ is open),

(ii) and the Beck-Chevalley condition.

Free geometric completion

It is relatively easy to construct a *free geometric completion* of a primary doctrine $P\colon \mathcal{C}^{op} \to \mathbf{MSLat}$.

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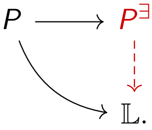
We first take the existential completion of P,



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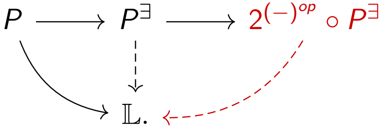
Given a primary doctrine morphism $P \to \mathbb{L}$ where $\mathbb{L}$ is also geometric.

We first take the existential completion of P, followed by the pointwise free join completion.



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To achieve idempotency, we must include relations.

For geometric logic, these relations look like Grothendieck topologies.

Doctrinal sites

Definition

A *doctrinal site* is a tuple $(\mathcal{C}, J, P, K)$ where

(i) $(\mathcal{C}, J)$ is a site,

(ii) $P\colon \mathcal{C}^{op} \to \mathbf{PreOrd}$ is a doctrine,

(iii) and K is a Grothendieck topology on the Grothendieck construction $\mathcal{C} \rtimes P$ that contains the *Giraud topology*.

Equivalently, $\pi\colon (\mathcal{C} \rtimes P, K) \to (\mathcal{C}, J)$ is a faithful fibration that is also a comorphism of sites.

Morphisms of doctrinal sites

Definition

A *morphism of doctrinal sites* $(\mathcal{C}, J, P, K) \to (\mathcal{D}, J', Q, K')$ consists of

(i) a functor $F \colon \mathcal{C} \to \mathcal{D}$,

(ii) and a natural transformation $a \colon P \to Q \circ F^{op}$,

such that both

$$F \colon (\mathcal{C}, J) \to (\mathcal{D}, J'), \quad F \rtimes a \colon (\mathcal{C} \rtimes P, K) \to (\mathcal{D} \rtimes Q, K')$$

are morphisms of sites.

Are doctrinal sites sane?

There are full and faithful embeddings

$$\mathbf{PrimDoc} \hookrightarrow \mathbf{DocSites}$$

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$$\mathbf{PrimDoc} \hookrightarrow \mathbf{DocSites}$$

$$\mathbf{ExDoc} \hookrightarrow \mathbf{DocSites}$$

given by

$$P \text{ existential doctrine} \mapsto (\mathcal{C}, J_{\text{triv}}, P, J_{\text{Ex}}),$$

where J_{Ex} is generated by covers

$$(c, U) \xrightarrow{f} (c, \exists_f U),$$

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There are full and faithful embeddings

$$\begin{array}{ccc} \mathbf{PrimDoc} & & \\ & \searrow & \\ \mathbf{ExDoc} & \hookrightarrow & \mathbf{DocSites} \\ & \nearrow & \\ \mathbf{CohDoc}, & & \end{array}$$

given by

$$P \text{ coherent doctrine} \mapsto (\mathcal{C}, J_{\text{triv}}, P, J_{\text{Coh}}),$$

where J_{Coh} is generated by covers

$$(c, U) \xrightarrow{f} (c, \exists_f U \vee \exists_g V) \xleftarrow{g} (c, V).$$

Geometric doctrines

Definition

A *geometric doctrine* over a site $(\mathcal{C}, J)$ is $\mathbf{Frm}_{\text{open}}$-valued doctrine $\mathbb{L} \colon \mathcal{C}^{op} \to \mathbf{Frm}_{\text{open}}$ satisfying one of the equivalent conditions

(i) $\mathbb{L}$ is an *internal locale* of $\mathbf{Sh}(\mathcal{C}, J)$,

(ii) the assignment of sieves $K_{\mathbb{L}}(d, V)$ given by

$$\left\{ (c_i, U_i) \xrightarrow{f_i} (d, V) \;\middle|\; i \in I \right\} \in K_{\mathbb{L}}(d, V) \iff V = \bigvee_{i \in I} \exists_{f_i} U_i$$

defines a Grothendieck topology on $\mathcal{C} \rtimes \mathbb{L}$ that contains the Giraud topology for J.

This is an application of Caramello's *relative Beck-Chevalley condition*.

The geometric completion

We define **GeomDoc** as the full subcategory of **DocSites** on objects of the form $(\mathcal{C}, J, \mathbb{L}, K_{\mathbb{L}})$.

Theorem

There exists a 2-adjunction, the *geometric completion*,

$$\mathbf{GeomDoc} \; \substack{\longleftarrow \\ \hookrightarrow} \; \mathbf{DocSites}.$$

Calculating the geometric completion

For the geometric morphism $C_\pi \colon \mathbf{Sh}(\mathcal{C} \rtimes P, K) \to \mathbf{Sh}(\mathcal{C}, J)$, the subobject lattice $\mathrm{Sub}_{\mathbf{Sh}(\mathcal{C} \rtimes P, K)}(C_\pi^*(F))$ is a frame for each object $F \in \mathbf{Sh}(\mathcal{C}, J)$.

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The doctrine

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The geometric completion acts on objects by sending $(\mathcal{C}, J, P, K)$ to the geometric doctrine

$$\mathcal{C}^{op} \xrightarrow{\ell^{op}} \mathbf{Sh}(\mathcal{C}, J)^{op} \xrightarrow{\mathrm{Sub}_{\mathbf{Sh}(\mathcal{C} \rtimes P, K)}(C_\pi^*(-))} \mathbf{Frm}.$$

In classical model theory, there is a notion of *elimination of imaginaries*.

Roughly speaking, an imaginary is a partial equivalence relation defined by a theory, and Shelah's elimination of imaginaries is a universal addition a sort for each such relation to the language of the theory.

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Makkai observed that the *pretopos completion* is the categorical formulation of Shelah's elimination of imaginaries for a coherent theory.

For doctrines, there are equivalent notions such as the *tripos-to-topos construction* or the *quotient completion*.

Syntactic categories

Let $\mathbb{L} \colon \mathcal{C}^{op} \to \mathbf{Frm}$ be a geometric doctrine over a cartesian base category.

Definition

The *syntactic category* $\mathbf{Syn}(\mathbb{L})$ of $\mathbb{L}$ is the category:

(i) whose objects are pairs (c, U), $U \in \mathbb{L}(c)$,

(ii) and whose arrows $(c, U) \xrightarrow{f} (d, V)$ are *provably functional relations*, i.e. $f \in \mathbb{L}(c \times d)$ such that

$$
\begin{aligned}
f(x) = y &\vdash_{x:c;y:d} U(x) \wedge V(y), \\
f(x) = y \wedge f(x) = y' &\vdash_{x:c;y,y':d} y = y', \\
U(x) &\vdash_{x:c} \exists y : d \; f(x) = y.
\end{aligned}
$$

Category of partial equivalence relations

Definition

The *category of partial equivalence relations* $\mathbf{PER}(\mathbb{L})$ of $\mathbb{L}$ is the category:

(i) whose objects are *partial equivalence relations* (c, E), i.e. $E \in \mathbb{L}(c \times c)$ such that

$$E(x, x') \vdash_{x,x':c} E(x', x),$$
$$E(x, x') \wedge E(x', x'') \vdash_{x,x',x'':c} E(x, x''),$$

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$$\begin{aligned} f(x) = y &\vdash_{x:c;y:d} E(x,x) \wedge F(y,y), \\ f(x) = y \wedge E(x,x') \wedge F(y,y') &\vdash_{x,x':c;y,y':d} f(x') = y, \\ f(x) = y \wedge f(x) = y' &\vdash_{x:c;y,y':d} F(y,y'), \\ E(x,x) &\vdash_{x:c} \exists y : d \; f(x) = y. \end{aligned}$$

Category of families of partial equivalence relations

Let κ be an infinite cardinal.

Definition

The *category of* κ*-families of partial equivalence relations* κ-$\mathbf{PER}(\mathbb{L})$ of $\mathbb{L}$ is the category:

(i) whose objects are tuples $(c_i, E_i)_{i \in I}$ of partial equivalence relations of at most length κ,

Category of families of partial equivalence relations

Let κ be an infinite cardinal.

Definition

The *category of κ-families of partial equivalence relations* κ-**PER**$(\mathbb{L})$ of $\mathbb{L}$ is the category:

(ii) and whose arrows $(c_i, E_i)_{i\in I} \xrightarrow{(f_{i,j})_{i\in I, j\in J}} (d_j, F_j)_{j\in J}$ are *familial provably functional relations*, i.e. $f_{i,j} \in \mathbb{L}(c_i \times d_j)$ such that

$$f_{i,j}(x) = y \vdash_{x:c_i;y:d_j} E_i(x,x) \wedge F_j(y,y),$$

$$f_{i,j}(x) = y \wedge E_i(x,x') \wedge F_j(y,y') \vdash_{x,x':c_i;y,y':d_j} f(x') = y,$$

$$f_{i,j}(x) = y \wedge f_{i,j}(x) = y' \vdash_{x:c_i;y,y':d_j} F_j(y,y'),$$

$$E_i(x,x) \vdash_{x:c_i} \bigvee_{j\in J} \exists y : d \; f_{i,j}(x) = y.$$

Families of partial equivalence relations and sheaves

The category ∞-**PER**($\mathbb{L}$) is the ∞-pretopos completion of **Syn**($\mathbb{L}$).

It is also equivalent to the *topos of internal sheaves* **Sh**($\mathbb{L}$) for $\mathbb{L}$ viewed as an internal locale. Explicitly,

$$\infty\text{-}\mathbf{PER}(\mathbb{L}) \simeq \mathbf{Sh}(\mathcal{C} \rtimes \mathbb{L}, K_{\mathbb{L}}).$$

This motivates considering the geometric morphism

$$\mathbf{Sh}(\mathcal{C} \rtimes P, K) \to \mathbf{Sh}(\mathcal{C}, J)$$

for a doctrinal site $(\mathcal{C}, J, P, K)$.

Summary of constructions

We thus obtain *generalised* exact completions of the geometric completion.

Level	Subcategory of $\mathbf{Sh}(\mathcal{C} \rtimes P, K)$	Corresponding completion
0	Full subcategory of subrepresentables	Syntactic category,
1	Full subcategory of sheaves covered by a subrepresentable	Exact completion,
ω_0	Full subcategory of sheaves covered by finitely many subrepresentables	Pretopos completion,
$\vdots$	$\vdots$	$\vdots$
∞	The whole topos	∞-pretopos completion.

Cioffo's strictification

Our framework also encompasses completions of a doctrine P that don't include any data coming from P.

For example, Cioffo studies *biased doctrines*, i.e. functors

$$P\colon \mathcal{C}^{op} \to \mathbf{MSLat}$$

where $\mathcal{C}$ has *weak finite limits*.

In order to study the quotient completion of a biased doctrine, parallel to Carboni and Vitale's exact completion of a weakly cartesian category, Cioffo introduces in [2] the *strictification* of a biased doctrine

$$P^{st}\colon \mathrm{fl}(\mathcal{C})^{op} \to \mathbf{MSLat}$$

where $\mathrm{fl}(\mathcal{C})$ denotes the free finite limit completion of $\mathcal{C}$.

Extending strictification

We can extend the notion of strictification to the geometric completion of any doctrinal site $(\mathcal{C}, J, P, K)$.

There exists a topology J' on $\mathrm{fl}(\mathcal{C})$ such that

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathrm{fl}(\mathcal{C}), J'),$$

and so we can define the *geometric strictification* of an arbitrary doctrinal site $(\mathcal{C}, J, P, K)$ as the doctrine

$$\mathrm{fl}(\mathcal{C})^{op} \xrightarrow{\ell^{op}} \mathbf{Sh}(\mathrm{fl}(\mathcal{C}), J') \simeq \mathbf{Sh}(\mathcal{C}, J) \xrightarrow{\mathrm{Sub}_{\mathbf{Sh}(\mathcal{C} \rtimes P, K)}(C_\pi^*(-))} \mathbf{Frm}.$$

Thank you for listening

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