Large Cardinals beyond Choice

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This is a report on joint work with

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Jensen's Dichotomy

Ronald Jensen proved in the early 70's that either V is very close to L, or very far from it. More precisely:

Theorem (R. Jensen)

Exactly one of the following hold:

1. *Every singular cardinal γ is singular in L, and $(\gamma^+)^L = \gamma^+$.*
2. *Every uncountable cardinal is inaccessible in L.*

Woodin's Dichotomy (Weak form)

A similar dichotomy for the model *HOD* was proved by Woodin a few years ago.

Theorem (W. H. Woodin)

Suppose that there is an extendible cardinal κ. Then exactly one of the following hold:

1. *Every singular cardinal $\gamma > \kappa$ is singular in HOD, and $(\gamma^+)^{HOD} = \gamma^+$.*
2. *Every regular cardinal $\gamma \geqslant \kappa$ is measurable in HOD.*

Two possible futures

The existence of large cardinals (e.g., $0^\sharp$) implies the second side of Jensen's dichotomy, whereas no traditional large cardinals imply that the second side of the HOD dichotomy must hold (because every traditional large cardinal is compatible with $V = HOD$).

Assuming the existence on an extendible cardinal we have two futures:

> **First future:** The first alternative of the HOD dichotomy holds and so HOD is close to V.
>
> **Second future:** The second alternative of the HOD dichotomy holds and HOD is far from V.

Which future will it be?

Two possible outcomes:

I **The Inner Model Program** (as outlined by Woodin) is successful, and then the first future must hold.

II **Large cardinals beyond Choice** are consistent with ZF, and a $0^\sharp$ analogue for HOD exists, and then the second future must hold.

The following results grew out of (failed) attempts to provide evidence for the first future, namely, of showing that some large cardinals beyond Choice are inconsistent with ZF.
Thus, the results reinforce the possibilities for the second future.

We investigate hierarchies of large cardinals that are inconsistent with the Axiom of Choice. The smallest of such cardinals were discovered by Reinhardt in 1965.

Reinhardt cardinals

A cardinal κ is Reinhardt if there is an elementary embedding

$$j : V \to V$$

with critical point κ.

Theorem (Kunen, 1971)

ZFC implies that Reinhardt cardinals don't exist. In fact, there is no non-trivial elementary embedding

$$j : V_{\lambda+2} \to V_{\lambda+2}$$

Reinhardt cardinals without Choice

Question

Is the existence of a Reinhardt cardinal consistent with ZF?

In order to make the problem easier, we study much stronger "Choiceless" large cardinal axioms.

The Reinhardt hierarchy

Definition

κ is super Reinhardt if for all ordinals λ there exists a non-trivial elementary embedding $j : V \to V$ such that $crit(j) = \kappa$ and $j(\kappa) > \lambda$.

If A is a proper class, then κ is A-super Reinhardt if for all ordinals λ there exists a non-trivial elementary embedding $j : V \to V$ such that $crit(j) = \kappa$, $j(\kappa) > \lambda$, and $j(A) = A$, where $j(A) := \bigcup_{\alpha \in OR} j(A \cap V_\alpha)$.

κ is totally Reinhardt if for each $A \in V_{\kappa+1}$,

$$\langle V_\kappa, V_{\kappa+1} \rangle \models ZF_2 + \text{"There is an } A\text{-super Reinhardt cardinal"}$$

A super Reinhardt strongly rank-reflects a Reinhardt

Theorem

If κ is super Reinhardt, then there exists $\gamma < \kappa$ such that

$$\langle V_\gamma, V_{\gamma+1}\rangle \models ZF_2 + \text{“There exists a Reinhardt cardinal”}$$

CON(totally R) $\Rightarrow$ CON(super R) $\Rightarrow$ CON(R)

The Berkeley hierarchy

If M is a transitive set, then let $\mathcal{E}(M)$ be the set of elementary embeddings $j : M \to M$.

Definition

δ is a proto-Berkeley cardinal if for every transitive set M such that $\delta \in M$ there exists $j \in \mathcal{E}(M)$ with $crit(j) < \delta$.

Theorem

If δ_0 is the least proto-Berkeley cardinal, then for all transitive sets M such that $\delta_0 \in M$ and all $\eta < \delta_0$ there exists $j \in \mathcal{E}(M)$ such that

$$\eta < crit(j) < \delta_0$$

Berkeley cardinals

Definition

δ is a Berkeley cardinal if for every transitive set M such that $\delta \in M$ and every $\eta < \delta$ there exists $j \in \mathcal{E}(M)$ with $\eta < crit(j) < \delta$.

Since being a Berkeley cardinal is a Π_2 property, no Σ_3 correct cardinals are less than or equal to the first Berkeley cardinal. In particular, if δ_0 is the least Berkeley cardinal, then there are no extendible cardinals $\leqslant \delta_0$.

A Berkeley strongly rank-reflects a Reinhardt

Theorem

If δ_0 is the least Berkeley cardinal, then there exists $\gamma < \delta_0$ such that

$$\langle V_\gamma, V_{\gamma+1}\rangle \models ZF_2 + \text{“There exists a Reinhardt card., witnessed by } j\text{,}$$

$$\text{and an } \omega\text{-huge card. above } \kappa_\omega(j)\text{”}$$

Club Berkeley cardinals

Definition

δ is a club Berkeley cardinal if δ is regular and for all clubs $C \subseteq \delta$ and all transitive sets M with $\delta \in M$ there exists $j \in \mathcal{E}(M)$ with $crit(j) \in C$.

Theorem

If δ is a club Berkeley cardinal, then it is totally Reinhardt.

Limit club Berkeley cardinals

Definition

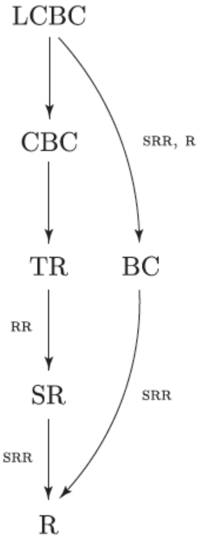
δ is a limit club Berkeley cardinal if it is a club Berkeley cardinal and a limit of Berkeley cardinals.

Theorem

If δ is a limit club Berkeley cardinal, then

$\langle V_\delta, V_{\delta+1}\rangle \models$ *"There is a Berkeley card. that is super Reinhardt"*

Implications



Many open questions

Questions

1. *Do totally Reinhardt cardinals strongly rank-reflect super Reinhardt cardinals?*
2. *Do super Reinhardt cardinals reflect Reinhardt cardinals?*
3. *Do Berkeley cardinals strongly rank-reflect super Reinhardt cardinals?*
4. *Do Berkeley cardinals rank-reflect totally Reinhardt cardinals?*

Gauging the failure of Choice

There is a tight connection between the amount of Choice one can have and the cofinality of the least Berkeley cardinal.

Theorem

If δ_0 is the least Berkeley cardinal, then $AC_{cof(\delta_0)}$ fails.

This raises an interesting question.

Question

What is the cofinality of the least Berkeley cardinal?

The cofinality of the least Berkeley cardinal

Theorem (B-K-W, R. Cutolo)

Assume ZF + There exists a Berkeley cardinal. Then there is a forcing extension that forces that the least Berkeley cardinal has cofinality ω.

Theorem (B-K-W, R. Cutolo)

Assume ZF + DC + There exists a Berkeley cardinal. Then there is an ω_1-preserving forcing extension that forces that the least Berkeley cardinal has cofinality ω_1.

Essentially the same proof works for other cofinalities.

A hierarchy of theories

The connection between the degree of Choice and the cofinality of the first Berkeley cardinal δ_0 suggest that the following hierarchy of theories has increasing strength.

$$ZF + BC + cof(\delta_0) = \omega$$

$$ZF + BC + AC_\omega + cof(\delta_0) = \omega_1$$

$$ZF + BC + AC_{\omega_1} + cof(\delta_0) = \omega_2$$

$$\vdots$$

$$ZF + BC + AC_{<\delta_0} + cof(\delta_0) = \delta_0$$

The HOD Dichotomy

Definition (Woodin)

An uncountable regular cardinal γ is ω-strongly measurable in *HOD* if there exists $\kappa < \gamma$ such that

1. $(2^\kappa)^{HOD} < \gamma$
2. There is no partition $\langle S_\alpha : \alpha < \kappa \rangle$ of $S^\gamma_\omega := \{\beta < \gamma : cof(\beta) = \omega\}$ into stationary sets such that $\langle S_\alpha : \alpha < \kappa \rangle \in HOD$.

Theorem (Woodin's HOD Dichotomy)

If there exists an extendible cardinal κ, then exactly one of the following hold:

1. *Every singular cardinal $\gamma > \kappa$ is singular in HOD and $(\gamma^+)^L = \gamma^+$.*
2. *Every regular cardinal $\geqslant \kappa$ is ω-strongly measurable in HOD.*

The Weak HOD Conjecture

Definition (The Weak HOD Conjecture)

The Weak *HOD* Conjecture is that

ZFC + "There is an extendible card. with a huge card. above it"

proves the *HOD* Hypothesis, i.e., there exists a proper class of regular cardinals that are not ω-strongly measurable in *HOD*.

Thus, if the Weak *HOD* Conjecture is true, then (assuming there is an extendible cardinal with a huge cardinal above it) the first alternative of the *HOD* Dichotomy must hold, and therefore V is close to *HOD*.

Evidence for the Conjecture comes from inner model theory.

Weak extender models for supercompactness

Definition (Woodin)

An inner model N of ZFC is a weak extender model for the supercompactness of κ if for every $\lambda > \kappa$ there exists a κ-complete normal fine measure U on $\mathcal{P}_\kappa(\lambda)$ such that

1. $N \cap \mathcal{P}_\kappa(\lambda) \in U$
2. $U \cap N \in N$.

Any reasonable inner model for a supercompact should satisfy those conditions.

The connection between weak extender model for supercompactness and the *HOD* Dichotomy is given by the following

Theorem (Woodin)

Suppose κ is an extendible cardinal. Then the following are equivalent:

1. *The HOD Hypothesis holds.*
2. *There is a regular cardinal $\gamma \geqslant \kappa$ which is not ω-strongly measurable in HOD.*
3. *No regular cardinal $\gamma \geqslant \kappa$ is ω-strongly measurable in HOD.*
4. *There is a cardinal $\gamma \geqslant \kappa$ such that $(\gamma^+)^{HOD} = \gamma^+$.*
5. *HOD is a weak extender model for the supercompactness of κ.*
6. *There is a weak extender model N for the supercompactness of κ such that $N \subseteq HOD$.*

Woodin's inner model program

Woodin's program for building an inner model for a supercompact cardinal hinges on proving the Ultimate-L Conjecture:

> *Suppose that κ is an extendible cardinal and there is a huge cardinal above κ. Then there exists a weak extender model N for the supercompactness of κ such that*
> 1. *N is weakly Σ_2-definable and $N \subseteq HOD$.*
> 2. *$N \models$ "$V =$ Ultimate-L".*

The Ultimate-L Conjecture implies the Weak HOD Conjecture. So, assuming an extendible cardinal with a huge cardinal above it, if the Ultimate-L Conjecture holds, then we are on the first side of the HOD Dichotomy, i.e., HOD is close to V.

Main theorem

Using some results of Woodin from his paper on Suitable Extender Models I[1], we proved the following

Theorem

Suppose that the Weak HOD Conjecture holds. Then there cannot be a non-trivial elementary embedding $j : V \to V$ along with an ω-huge cardinal above $\kappa_\omega(j)$.

Corollary

Assume the Weak HOD Conjecture holds. Then there are no Berkeley cardinals.

Thus, the Ultimate-L Conjecture (and even the Weak HOD Conjecture) wipes out almost all of the hierarchy of large cardinals beyond Choice (e.g., super Reinhardt and Berkeley cardinals).

[1] Journal of Mathematical Logic, 10 (1-2), 101-339, June 2010.

The second future: Large Cardinals beyond Choice

If large cardinals beyond Choice are consistent with ZF, then the Weak *HOD* Conjecture fails, and the Ultimate-*L* Conjecture also fails.

Note: this does not strictly imply that we must be on the second side of the *HOD* Dichotomy.

However, assuming there exists an extendible cardinal, if there is, e.g., a *HOD* analogue of a Berkeley cardinal (now in a ZFC setting), then we must be on the second side of the *HOD* Dichotomy, where *HOD* is far from V.