Describing limits of bounded sequences of measurable functions via nonstandard analysis

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XXVI incontro AILA - Padova

Different notions of limit - I

The Rademacher sequence

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$$r_0(x) = \chi_{[0,1/2)}(x \bmod 1) - \chi_{[1/2,1)}(x \bmod 1)$$

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- $\lim_{n\to\infty} r_n(x)$ does not exist for a.e. $x \in [0,1]$;
- $r_n \rightharpoonup 0$ ($r_n \overset{\star}{\rightharpoonup} 0$ if $p = 1, \infty$);

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- for all $f \in C^0(\mathbb{R})$ and for all $\varphi \in C^0([0,1])$,

$$\lim_{n\to\infty} \int_{[0,1]} f(r_n(x))\varphi(x)dx = \left(\frac{1}{2}f(1) + \frac{1}{2}f(-1)\right)\int_{[0,1]} \varphi(x)dx.$$

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Theorem (The main theorem of Young measures)

For every bounded sequence of $L^p(\Omega)$ functions $\{z_n\}_{n\in\mathbb{N}}$, there exists a Young measure ν such that for all $f \in C^0_b(\mathbb{R})$ and for all $\varphi \in C^0([0,1])$,

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$$z_n \xrightarrow{Y} \nu.$$

Nonstandard analysis in a nutshell

Let $\mathcal{P}(X)$ be the power set of X, and define

$$\mathbb{V}(X) = \bigcup_{n\in\mathbb{N}} \mathcal{P}^n(X).$$

Definition (Nonstandard universe)

A nonstandard universe is a triple $(\mathbb{V}(\mathbb{R}), \mathbb{V}({}^*\mathbb{R}), {}^*)$ such that:

- ${}^* : \mathbb{V}(\mathbb{R}) \to \mathbb{V}({}^*\mathbb{R})$;
- * maps $\mathbb{R}$ properly into ${}^*\mathbb{R}$ (i.e. $\mathbb{R} \neq {}^*\mathbb{R}$);
- * preserves "elementary properties".

The basic idea of nonstandard analysis

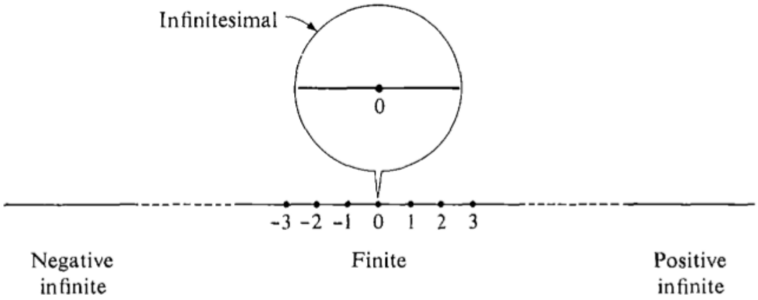

Image from "Elementary Calculus: An Infinitesimal Approach", © 2000 by H. Jerome Keisler.

Grid functions

Definition (Grid functions)

Let $N \in {}^*\mathbb{N}$ be infinite and let $\varepsilon = N^{-1}$. We define

$$\mathbb{X} = \{n\varepsilon : n \in {}^*\mathbb{Z} \text{ and } -N^2 \leq n \leq N^2\}.$$

The space of grid functions defined over an open set $\Omega \subseteq \mathbb{R}^k$ is

$$\mathbb{G}(\Omega) = \{f : {}^*\Omega \cap \mathbb{X}^k \to {}^*\mathbb{R}\}.$$

The main theorem

Theorem

For every bounded sequence $\{z_n\}_{n\in\mathbb{N}}$ in $L^p(\Omega)$, there exists a (non unique) function $z \in \mathbb{G}(\Omega)$ such that

- *for all $\varphi \in C^0(\Omega)$ it holds*

$$\varepsilon^k \sum_{x \in {}^*\Omega \cap \mathbb{X}^k} z(x)^* \varphi(x) \approx \lim_{n\to\infty} \int_\Omega z_n(x)\varphi(x)dx;$$

- *for all $f \in C^0_b(\mathbb{R})$ and for all $\varphi \in C^0(\Omega)$ it holds*

$$\varepsilon^k \sum_{x \in {}^*\Omega \cap \mathbb{X}^k} {}^*f(z(x))^* \varphi(x) \approx \lim_{n\to\infty} \int_\Omega f(z_n(x))\varphi(x)dx.$$

Idea of the proof - I

Theorem

For every distribution T, there exists $u \in \mathbb{G}(\Omega)$ that satisfies

$$\varepsilon^k \sum_{x \in {}^*\Omega \cap \mathbb{X}^k} u(x)^* \varphi(x) \approx \langle T, \varphi \rangle \text{ for all } \varphi \in C_c^\infty(\Omega).$$

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Theorem (Cutland – 1986)

For every Young measure ν, there exists $u \in \mathbb{G}(\Omega)$ that satisfies

$$\varepsilon^k \sum_{x \in {}^*\Omega \cap \mathbb{X}^k} {}^*f(u(x))^*\varphi(x) \approx \int_\Omega f(\nu(x))\varphi(x)dx$$

for all $f \in C_b^0(\mathbb{R})$ and for all $\varphi \in C^0(\Omega)$

Idea of the proof - II

The ingredients of the grid function z

- Let $z_n \rightharpoonup z_\infty$ ($z_n \overset{\star}{\rightharpoonup} z_\infty$ if $p = 1, \infty$).

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- Let $z_Y \in \mathbb{G}(\Omega)$ correspond to the Young measure ν.

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Theorem

$z = z_D + z_Y \in \mathbb{G}(\Omega)$ *satisfies the desired properties.*

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Perspectives for future research

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Possible applications of grid functions include:

- the definition of new notions of limits;
- the description of physical phenomena that cannot be formalized in the space of distributions;
- the study of generalized solutions to classically ill-posed problems from many areas of functional analysis.