Left distributive algebras beyond I0

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Forget about large cardinals.

Question

Let V_κ the cumulative hierarchy of sets. Is there a non-trivial elementary embedding $j : V_\eta \prec V_\eta$?

There are some limitations:

In these cases, if j is not trivial, then some ordinals are moved. We call *critical point* of j the least ordinal (cardinal) moved.

Let $\kappa_0 = \mathrm{crt}(j)$. We can define $\kappa_{n+1} = j(\kappa_n)$, and $\lambda = \sup_{n\in\omega} \kappa_n$ (this is called the *critical sequence*.

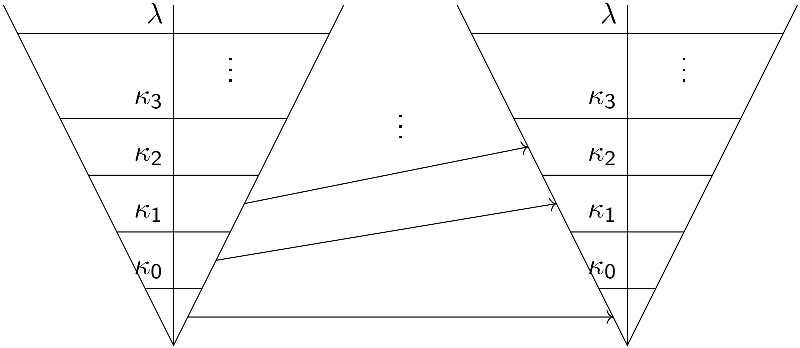
Theorem (Kunen)

If $j : V_\eta \prec V_\eta$ and there is a well-ordering of $V_{\lambda+1}$ in V_η, then $1 = 0$.

So η can only be limit or successor of limit.

Assumption

I3: There are elementary embeddings $j : V_\lambda \prec V_\lambda$, λ limit.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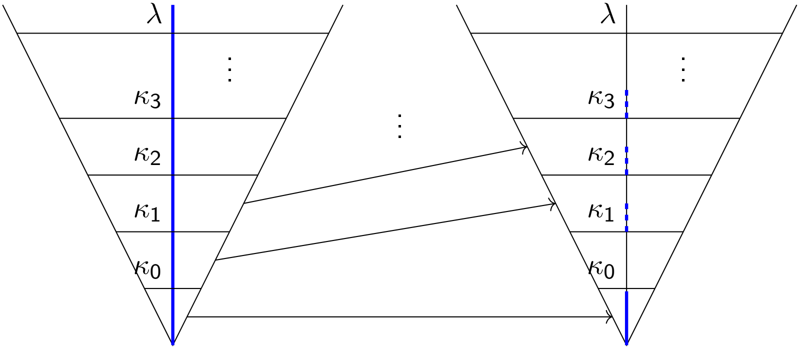
λ
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We can extend a little bit the scope of j.

Picture: slicing a subset of V_λ.

Lemma

Let $j : M \prec N$. Let $X \subseteq M$. Suppose that:

- $M \cap Ord$ and $N \cap Ord$ are singular cardinals;
- j is cofinal;
- X is amenable, i.e., rank-fragments of X are in M.

Then $j^+ : (M, X) \prec (N, j^+(X))$.

Special case: $X = k : V_\lambda \prec V_\lambda$. Therefore $j^+(k) : V_\lambda \prec V_\lambda$. We write $j \cdot k$.

This is not to be confused with $j \circ k$! For example:

- critical sequence of $j \circ j$: $\kappa_0, \kappa_2, \kappa_4, \ldots$
- critical sequence of $j \cdot j$: by elementarity $\mathrm{crt}(j(j)) = j(\mathrm{crt}(j))$, so $\kappa_1, \kappa_2, \kappa_3 \ldots$

This is an operation on the space $\mathcal{E}_\lambda = \{j : V_\lambda \prec V_\lambda\}$, called *application*. What is its algebra? What are the rules?

Keep in mind that $j(k)$ is difficult to calculate: while, for example, $j \circ k(x)$ is definable from j, k, x, this is not true for $j \cdot k(x)$, that is known only on $\mathrm{ran}(j)$.

One rule is left-distributivity:

$$j \cdot (k \cdot l) = (j \cdot k) \cdot (j \cdot l)$$

so $(\mathcal{E}_\lambda, \cdot)$ is a left distributive algebra. Are there other rules?

Let T_n be the sets of words constructed using generators $x_1, \ldots, x_n$ and the binary operator $\cdot$.

Let $\equiv_{LD}$ the congruence on T_n generated by all pairs of the form $t_1 \cdot (t_2 \cdot t_3), (t_1 \cdot t_2) \cdot (t_1 \cdot t_3)$. Then $T_n / \equiv_{LD}$ is the *universal free* LD-algebra with n generators. We call it F_n.

Given an LD-algebra A, we can consider its subalgebra A_X generated by the elements in a finite subset X. There is always a surjective homomorphism from $F_{|X|}$ to A_X. We say that A_X is *free* if it is an isomorphism.

In other words, A_X is free iff if two elements of A_X are equal, it must be because of left-distributivity.

Theorem (Laver)

Let $j; V_\lambda \prec V_\lambda$. Then $\mathcal{E}_{\{j\}}$ is free.

Open problem

What about $A_{\{j,k\}}$? Can it be free?

This is a hard problem. We have to prove many inequalities at the same time, and since an embedding can be represented by many words there is no clear order to use induction.

For the one generator case this was useful, and still holds:

Theorem (Laver, Steel)

Let $\leq_L$ be the left-division, i.e., $w <_L v$ iff there are $u_1, \dots u_n$ such that $v = (\dots((w \cdot u_1) \cdot u_2) \cdot \dots \cdot u_n)$.
Then $<_L$ is irreflexive on $\mathcal{E}_\lambda$.

By Laver's Criterion, this is enough to prove freeness for one generator. For the many-finite-generators case, there is the Dehornoy's Criterion, that wants irreflexivity that wants $(\dots((((c_1 \cdot \dots) \cdot c_r) \cdot x) \cdot a_1) \dots) \cdot a_p \neq$ $(\dots((((c_1 \cdot \dots) \cdot c_r) \cdot y) \cdot b_1) \dots) \cdot b_q$ for any c's, a's. b's and x, y different generators.

Let's change perspective, and widen our horizons.

A $j : V_\lambda \prec V_\lambda$ is a Σ^1_0-elementary embedding (or, equivalently, j^+ is Σ_0-elementary). What if it is completely second-order, i.e. $j : V_{\lambda+1} \prec V_{\lambda+1}$? This is called I1. What is the algebra there?

Well, how to define application? $j(k) = (j(k \upharpoonright V_\lambda))^+$.

Lemma (Laver)

If $j, k : V_{\lambda+1} \prec V_{\lambda+1}$, then $j(k) : V_{\lambda+1} \prec V_{\lambda+1}$.

But then $\rho : \mathcal{E}^+_{\{j_1,\dots,j_n\}} \to \mathcal{E}_{\{j_1,\dots,j_n\}}$, $\rho(j) = j \upharpoonright V_\lambda$ is an isomorphism.

So the freeness of subalgebras of I3 and I1 embeddings is the same problem. We would like to see genuinely new structures. Let's go up.

Definition

If X is a set, then $L(X)$ is the smallest ZF-model that contains X. It is the class of the sets constructible from X.

I0 is the existence of an embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, with $\mathrm{crt}(j) < \lambda$. How to define the application here?

Not clear at all. The problem is that j is not amenable in $L(V_{\lambda+1})$! In fact, there is a Θ such that $j \upharpoonright L_\Theta(V_{\lambda+1}) \notin L(V_{\lambda+1})$. So we cannot play the game of cutting and reassembling.

Every element of $L(V_{\lambda+1})$ is definable from an ordinal and an element of $V_{\lambda+1}$. So the behaviour of j depends only on its behaviour on $Ord \times V_{\lambda+1}$. We can actually split the two:

Theorem (Woodin)

Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$. Then there are two embeddings $j_U, k_U : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ such that $j = k_U \circ j_U$ and

- $\mathrm{crt}(j_U) < \lambda$ and it comes from an ultrafilter, so its behaviour it's definable from $j_U \upharpoonright V_\lambda$;
- $k_U(X) = X$ for any $X \subseteq V_{\lambda+1}$.

We say that j is *weakly proper* if $j = j_U$. We can define now application on weakly proper embeddings, with $j \cdot k$ the only embedding that can extend $j(k \upharpoonright V_\lambda)$.

Theorem (Woodin)

If $j, k : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ are weakly proper, then $j \cdot k : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ and it is weakly proper.

We will see that the function that associates a weakly proper embedding to its restriction on V_λ is an isomorphism. Nothing new here. Let's go up again.

Without going into much detail, $(V_{\lambda+1})^\sharp$ is some description of the theory of $L(V_{\lambda+1})$ that is codeable as a subset of $V_{\lambda+1}$.

We can consider elementary embeddings $j : L(V_{\lambda+1}, (V_{\lambda+1})^\sharp) \prec L(V_{\lambda+1}, (V_{\lambda+1})^\sharp)$. But again, isomorphism.

Up again: $(V_{\lambda+1})^{\sharp\sharp}$, $(V_{\lambda+1})^{\sharp\sharp\sharp}$, ..., at the union we unite everything and close by construction(over-over-simplifying). This is the E_α hierarchy, its components are $L(E_\alpha)$.

Theorem (Woodin)

Let $j : L(E_\alpha) \prec L(E_\alpha)$ with $\mathrm{crt}(j) < \lambda$. Then there are two embeddings $j_U, k_U : L(E_\alpha) \prec L(E_\alpha)$ such that $j = k_U \circ j_U$ and

- $\mathrm{crt}(j_U) < \lambda$ and it comes from an ultrafilter, so its behaviour it's definable from $j_U \upharpoonright E_\alpha$;
- $k_U(X) = X$ for any $X \in E_\alpha$.

For an initial segment of this hierarchy, we have $L(E_\alpha) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$. We are going to define application for $j, k : L(E_\alpha) \prec L(E_\alpha)$ such that

- j, k are weakly proper;
- $L(E_\alpha) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$.

Again, j, k are not amenable, but we can fragment them in another way. If j, k are weakly proper, then there is a class of cardinals I that are fixed points for j and k. For any $s \in I^{<\omega}$, β "small enough", we consider the following fragments:

$$Z^s_\beta = H^{L(E_\alpha)}(s \cup V_{\lambda+1} \cup \{V_{\lambda+1}\} \cup \{E_\alpha\} \cup \beta$$

Now we consider $j(k \restriction Z^s_\beta) : Z^s_{j(\beta)} \prec Z^s_{j(k(\beta))}$. Then $\bigcup_{s\in I^{<\omega},\beta\in\mathrm{Ord}} Z^s_{j(\beta)} \prec L(V_{\lambda+1})$ and $\bigcup_{s\in I^{<\omega},\beta\in\mathrm{Ord}} Z^s_{j(k(\beta))} \prec L(V_{\lambda+1})$.

In other words: rank-by-rank the embeddings are not amenable, but we can fragment an elementary submodel of $L(V_{\lambda+1})$ on pieces over which the embeddings are amenable. Between pairs of fragments we have elementary embeddings that are coherent, so $j \cdot k$ will be the direct limit of the directed system of the $j(k \restriction Z^s_\beta)$.

Theorem

Let $\mathcal{E}(E_\alpha)$ be the "set" of weakly proper elementary embeddings from $L(E_\alpha)$ to itself. Then $(\mathcal{E}(E_\alpha), \cdot)$ is a left distributive algebra.

Remark

Let $\rho : \mathcal{E}(E_\alpha) \to \mathcal{E}_\lambda$ the restriction to V_λ. Then ρ is a homomorphism.

Proposition

Let $\mathcal{E}_j$ be the subalgebra generated by $\{j\}$. Then $\mathcal{E}_j$ is free.

Proof

The following diagram commutes.

$$\begin{tikzcd} F_1 \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & \mathcal{E}_j \arrow[d, "\rho"] \\ & \mathcal{E}_{\rho(j)} \end{tikzcd}$$

Therefore $\rho : \mathcal{E}_j \to \mathcal{E}_{\rho(j)}$ is an isomorphism.

Remark

Suppose that for any $j, k : L(E_\alpha) \prec L(E_\alpha)$ weakly proper $j = k$ iff $j \restriction V_\lambda = k \restriction V_\lambda$. (For example for $\alpha < \lambda$). Then $\rho : \mathcal{E}_{j,k} \to \mathcal{E}_{\rho(j),\rho(k)}$ is an isomorphism.

Will we ever find a genuinely new algebra?

For this, we have to go from the temperate zone to higher altitude, the habitat of the strange creatures known as non-proper elementary embeddings.

Definition

An embedding $j : L(E_\alpha) \prec L(E_\alpha)$ is proper iff it is weakly proper and for every $X \in E_\alpha$, $\langle X, j(X), j^2(X), \dots \rangle \in L(E_\alpha)$.

Remark

- Every $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ that is weakly proper is proper;
- If α is a successor cardinal, every $j : L(E_\alpha) \prec L(E_\alpha)$ that is weakly proper is proper;
- If α is a limit cardinal of cofinality $> \omega$, every $j : L(E_\alpha) \prec L(E_\alpha)$ that is weakly proper is proper;
- If α is small (say $\alpha < \lambda$), every $j : L(E_\alpha) \prec L(E_\alpha)$ that is weakly proper is proper.

Theorem

If the E_α hierarchy is "long enough", there there is an α such that:

- $L(E_\alpha) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$;
- there exist $j : L(E_\alpha) \prec L(E_\alpha)$ and $k : L(E_\alpha) \prec L(E_\alpha)$ such that j, k are weakly proper, j is proper, k is not proper and $j \restriction V_\lambda = k \restriction V_\lambda$.

This is it! This is finally a different algebra! Now ρ is still a homomorphism, but it is not an isomorphism.

This is fodder for many new inequalities, and some even meet Dehornoy's criterion!

There are three different kinds of inequalities:

- Laver-Steel Theorem, that holds because ρ is an homomorphism. So $j \neq j \cdot k$, $j \cdot k \neq (j \cdot k) \cdot j$, ...
- Actually $\rho : \mathcal{E}_{j,k} \to \mathcal{E}_{\rho(j)}$, that is free, so $j \neq k \cdot j$, $j \cdot j \neq (j \cdot k) \cdot j$, ...
- By elementarity, properness is preserved, so $j \cdot k \neq k \cdot j$, $(j \cdot k) \cdot j \neq (j \cdot k) \cdot j$ ...

Unfortunately some inequalities from Dehornoy's criterion do not fall in these rules: Is $j \cdot k \neq k \cdot k$?

So this leaves us with the question:

Open problem

Is $\mathcal{E}_{j,k}$ free?

Thanks you for your attention