EUCLIDEAN NUMBERS,
EUCLIDEAN NUMEROSITY,
AND EUCLIDEAN PROBABILITY

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The field of the Euclidean numbers

From the algebraic point of view, the Euclidean numbers are a non-Archimedean ordered superfield of the reals $\mathbb{E}$, with a supplementary structure, the Euclidean structure, introduced axiomatically by the operation of (ordinal-indexed) transfinite sum:

$$\sum_{k<\alpha} a_k$$

where the $a_k$'s are real numbers, while $k$ and $\alpha$ are ordinals smaller than the first inaccessible ordinal $\Omega$.

We assume that a transfinite sum coincides with the ordinary sum of the field $\mathbb{E}$ if the number of non-zero summands is finite.
Let \( S(\Omega, E) = \{ \xi \in E^\Omega \mid \exists j \in \Omega, \forall \beta > j, \xi_j = 0 \} \)
be the set of all \textit{eventually 0} \( \Omega \)-sequences of elements of \( E \), and let \( \Sigma : S(\Omega, E) \rightarrow E \) be an \( E \)-linear map.

For \( \xi = \langle \xi_k \mid k \in \Omega \rangle \in S(\Omega, E) \) define \( \Sigma(\xi) := \sum_k \xi_k \),
(the \textit{transfinite sum of the Euclidean numbers} \( \xi_k, k \in \Omega \)).

We give three natural axioms that rule the behaviour of these transfinite sums.

\textbf{RA \ Real numbers Axiom:}

\textit{For all} \( \xi \in E \text{ there exists} \ x \in S(\Omega, \mathbb{R}) = S(\Omega) \cap \mathbb{R}^\Omega \text{ such that} \}
\[ \Sigma(x) = \xi \]
Grounding on the axiom RA, in the following axioms we restrict ourselves to considering transfinite sums of real numbers.

For sake of clarity, we denote general Euclidean numbers by greek letters \( \xi, \eta, \zeta \), and real numbers by latin letters \( x, y, z \).

**DA Double sum axioms:**

**DA1** \[
\sum_h \sum_k x_{hk} = \sum_k \sum_h x_{hk} = \sum_i \sum_{h \lor k = i} x_{hk}.
\]

So we may denote any of these double sums by \( \sum_{h,k} x_{hk} \).

**DA2** Let \( x_{hk} = 0 \) for \( h, k \geq \omega^j \). Then
\[
\sum_{h,k} x_{hk} = \sum_i y_i \quad \text{where} \quad y_i = \begin{cases} x_{hk} & \text{if} \quad i = \omega^j \odot h + k \\ 0 & \text{otherwise} \end{cases}.
\]
CA Comparison Axiom:

For all \( x, y \in S(\Omega) \cap \mathbb{R}^\Omega \), if there exists \( \beta \in \Omega \) such that

\[
\sum_{k \sqsubseteq \alpha} x_k \leq \sum_{k \sqsubseteq \alpha} y_k \quad \text{for all } \alpha \supseteq \beta,
\]

then

\[
\sum_k x_k \leq \sum_k y_k.
\]

The formal inclusion \( \sqsubseteq \) between ordinals is a lattice partial ordering extending the natural ordering, such that

\[
\{ \xi \mid \xi \sqsubseteq \alpha \} \text{ is finite for all } \alpha \in \Omega
\]

It is defined according to the normal forms:

\[
\alpha = \sum_{n=1}^{N} \omega^j n \odot a_n \sqsubseteq \beta = \sum_{n=1}^{N} \omega^j n \odot b_n \iff a_n \leq b_n \quad \text{for } 1 \leq n \leq N.
\]

So all sums \( \sum_{k \sqsubseteq \alpha} \) are ordinary finite sums of real numbers.

* The name formal inclusion should recall that the respective coefficient free normal forms, considered as multisets, are indeed contained one inside of the other one.
These simple and natural axioms endow $\mathbb{E}$ with a very rich structure. First a few simple consequences.

- **Product formula:**

  $$(\sum_{h} x_h)(\sum_{k} y_k) = \sum_{i} \sum_{h \lor k = i} x_h y_k.$$  

- **Translation invariance:** Let $x_k = 0$ for $k \geq \omega^j$. Then

  $$\sum_{k} x_k = \sum_{i} y_i, \quad \text{where} \quad y_i = \begin{cases} x_k & \text{if } i = \omega^j \odot h + k \\ 0 & \text{otherwise} \end{cases}.$$  

- **Associativity:**

  Put $\sum_{k \in [\alpha, \beta]} \xi_k = \sum_{k} \xi_k (\chi_{\beta}(k) - \chi_{\alpha}(k))$. Then

  $$\sum_{k \in [0, \omega^j \odot \beta]} x_k = \sum_{h \in [0, \beta]} \sum_{k \in [\omega^j \odot h, \omega^j \odot (h+1)]} x_k.$$
The theory of the Euclidean numbers combines the Cantorian theory of ordinal numbers with Non Standard Analysis (NSA).

1. Every Euclidean number can be obtained as a transfinite sum of real numbers; a transfinite sum of Euclidean numbers is well defined, and can be obtained as *limit of ordinal-indexed (finite) partial sums*, when suitable topologies are given to $\mathbb{E}$ and to the ordinals.

2. Any accessible ordinal $\alpha \in \Omega$ can be identified with the transfinite sum of $\alpha$ ones in $\mathbb{E}$; this identification is consistent with the *natural ordinal operations* $\oplus$ and $\odot$, so $\mathbb{E}$ is a sort of *natural* extension of the *semiring* of the ordinal numbers in $\Omega$. 
3. The Euclidean numbers are a *hyperreal field*; more precisely \( \mathbb{E} \) is the *unique saturated real closed field having the cardinality of* \( \Omega \). It is isomorphic to the hyperreal Keisler field [keisler76], and every ordered field of accessible cardinality is (isomorphic to) a subfield of \( \mathbb{E} \).

The ordinals as embedded in \( \mathbb{E} \) are a (proper) subsemiring of the hypernatural numbers \( \ast \mathbb{N} \) of \( \mathbb{E} \).

4. The Euclidean numbers are strictly related to the notion of *numerosity*, introduced by Benci, Di Nasso and Forti in order to save the five Euclidean common notions on *magnitudines*. In fact, \( \mathbb{E} \) can be charachterized as the hyperreal field generated by the real numbers and the semiring of numerosities
We have chosen to call $\mathbb{E}$ the field of the *Euclidean* numbers for two main reasons: firstly, this field arises in a numerosity theory (including all subsets of $\Omega$), which saves all the Euclidean common notions, including the fifth

*The whole is greater than the part,*

in contrast to the Cantorian theory of cardinal numbers.

Secondly, the field $\mathbb{E}$ describes the Euclidean continuum better than the *real field* $\mathbb{R}$, at least when considering the *set theoretic reduction* of $\mathbb{R}$. This last point has been extensively dealt with in [Benci-Freg.]. As a simple example, note that, contrary to the Euclidean requirement, a segment cannot be divided by a point into two congruent pieces.
The notion of numerosity

A measure of size for arbitrary sets should abide by the famous five common notions of Euclid's Elements, which traditionally embody the properties of any kind of magnitudes,

1. Things equal to the same thing are also equal to one another.
2. And if equals be added to equals, the wholes are equal.
3. And if equals be subtracted from equals, the remainders are equal.
4. Things [exactly] applying onto one another are equal to one another.
5. The whole is greater than the part.

NB Translating \(\epsilonφαρμοζοντα\) by “applying [exactly] onto”, instead of the usual “coinciding with” seems to give a more appropriate rendering of the Euclidean usage of the verb \(\epsilonφαρμωζειν\), which refers to superposition of congruent figures.

How to save the Euclidean common notions?

It is worth noticing that traditional geometry satisfies the Euclidean common notions because there is a restricted class of “exact applications” (e.g., the rigid equidecompositions of polygons). So the question arises as to which correspondences can be taken as “exact applications” in order to fulfill the five Euclidean common notions.

Cantor himself, besides cardinals based on general bijections, introduced ordinal numbers, assigned to sets endowed with a wellordering, restricting the “exact applications” to the order preserving bijections. But the Euclid’s principle EP still badly fails.
The *Euclidean* numbers are an *extension* of the ordinals, suitable to provide a notion of size satisfying *all* the Euclidean common notions for an appropriate class of “*labelled sets*”. A labelled set can be viewed as a generalization of a wellordered set, because the latter can be naturally labelled by the unique order isomorphism with an (initial segment of an) ordinal. In fact it will turn out that any labelled set is “equinumerous” to a *set of ordinals*. The idea is that by putting an appropriate *labelling* on arbitrary sets, the *label preserving bijections* (intended as “exact applications”) might produce the “nonnegative integers” of the Euclidean numbers (whence their name).
Labelled sets

A \textit{labelled set} is a pair \((E, \ell)\), where

- \(E\) is a set of cardinality less than the first inaccessible \(\Omega\);
- \(\ell : E \to \Omega\) is a function (the \textit{labelling function}) such that
  1. the set \(\ell^{-1}(x)\) is finite for all \(x \in \Omega\);
  2. \(\ell(x) = x\) for all \(x \in E \cap \Omega\).

- Two labelled sets \((E_1, \ell_1)\) and \((E_2, \ell_2)\) are \textit{isomorphic} if there is a \textit{biunique} map \(\phi : E_1 \to E_2\) such that
  \[
  \ell_2(\phi(x)) = \ell_1(x) \quad \text{for all } x \in E_1.
  \]

- Two labelled sets \((E_1, \ell_1)\) and \((E_2, \ell_2)\) are \textit{coherent} if
  \[
  \ell_1(x) = \ell_2(x) \quad \text{for all } x \in E_1 \cap E_2.
  \]
• The **basic operations** on labelled sets are the following:

1. **Subset** - A subset of a labelled set \((E, \ell)\) is a labelled set \((F, \ell|_F)\) where \(F \subseteq E\);

2. **Union** - The union of two coherent labelled sets \((E_1, \ell_1)\), \((E_2, \ell_2)\) is the labelled set \((E_1 \cup E_2, \ell)\) where \(\ell(x) = \begin{cases} \ell_1(x) & \text{if } x \in E_1 \\ \ell_2(x) & \text{if } x \in E_2 \end{cases}\)

3. **Cartesian product** - The Cartesian product of two labelled sets \((E_1, \ell_1)\), \((E_2, \ell_2)\) is the labelled set \((E_1 \times E_2, \ell)\) where \(\ell(x_1, x_2) = \ell_1(x_1) \lor \ell_2(x_2)\)

• A family \((\mathcal{A}, \ell)\) of pairwise coherent labelled sets is **closed** if it is closed under the three basic operations.
A *Euclidean numerosity theory* is a pair \((\mathbb{U}, n)\), where

- \(\mathbb{U}\) is a *closed family* of labelled set, and
- \(n : \mathbb{U} \rightarrow \mathbb{E}\) is the *Euclidean numerosity function* associating to each labelled set \((E, \ell) \in \mathbb{U}\) its euclidean numerosity
  \[
  n(E, \ell) = \sum_k |\ell^{-1}(k)| \in \mathbb{E};
  \]
- The set of numerosities of \((\mathbb{U}, n)\) is the range \(\mathcal{N} = n[\mathbb{U}] \subseteq \mathbb{E}\).

Remark Given a coherent family \((\mathcal{A}, \ell)\) of labelled sets, there exists a *least closed family* \(\mathbb{U}(\mathcal{A}, \ell)\) including \(\mathcal{A}\), called the *closure of\((\mathcal{A}, \ell)\).* We omit the labelling function \(\ell\) when it is clear from the context.
Canonical examples

Recall that we identify the natural numbers with the finite ordinal numbers, and the accessible ordinals with the corresponding Euclidean numbers.

• Let $F \subset \Omega$ be finite. Then $\mathbb{U}(\{F\})$ contains only finite sets, so $n$ is the finite cardinality, and $\mathbb{N} = n[\mathbb{U}(\{F\})] = \mathbb{N} \subset \mathbb{E}$.

• $(\mathbb{U}[\mathbb{N}], n)$ is the “simplest” numerosity theory containing infinite sets, and $\mathbb{N} = n[\mathbb{U}(\mathbb{N})] \subset *\mathbb{N} \subset \mathbb{E}$.

• The canonical numerosity theory $(\mathbb{U}(\Omega), n)$ is the “simplest” theory which contains all the (accessible) ordinal numbers, and $\Omega \subset \mathbb{N} = n[\mathbb{U}(\Omega)] \subset *\mathbb{N} \subset \mathbb{E}$. 
Main properties of the Euclidean numerosities

Theorem Let \((\mathbb{U}, n)\) be a Euclidean numerosity theory. Then one has, for all \(A, B \in \mathbb{U}\):

**Sum-Difference:** \(n(A \cup B) = n(A) + n(B) - n(A \cap B)\);

**Part-Whole:** \(A \subset B \implies n(A) < n(B)\);

**Cartesian Product:** \(n(A \times B) = n(A) \cdot n(B)\);

**Comparison:** if the 1-to-1 map \(T : A \to B\) preserves labels, then \(n(A) \leq n(B)\).

Moreover, if \(\Omega \subseteq A\), then

- each (accessible Von Neumann) ordinal is its own numerosity,
- each set in \(\mathbb{U}\) is equinumerous to a set of ordinals.
Saving the five Euclidean common notions

The Euclidean numerosities satisfy all the five Euclidean common notions, when interpreted in the natural way:

1. Two labelled sets are equal (in size) if they have the same numerosity;
2. The addition of two (disjoint) labelled sets is given by their union;
3. The subtraction of a labelled set from a (coherent) superset is given by relative complement;
4. Two labelled sets (exactly) apply onto one another if they are isomorphic;
5. A part of a labelled set is just a (coherent) subset.
Other properties of the Euclidean numerosities

Translation invariance: \( \forall E \subseteq \omega^j, \ n(\{\omega^j + \xi \mid \xi \in E\}) = n(E); \)

Homothety invariance: \( \forall E \subseteq \omega^\omega j, \ n(\omega^\omega j \xi \mid \xi \in E) = n(E); \)

Cartesian product: \( \forall \alpha, \beta \in \Omega, \ n(\alpha \times \beta) = \alpha \beta; \)

Label the set \( P_\omega(E) \) of the finite parts of the labelled set \( (E, \ell) \) by the label \( \lor \ell \) such that \( \lor \ell (\{a_1, \ldots, a_n\}) = \lor_{k=1}^n \ell(a_k). \)

If the family \( U \) is closed under the operation \( P_\omega \), one has

Finite parts: \( n(P_\omega(E)) = 2^{n(E)}. \)

† Notice that the product of the field \( E \) agrees with the natural product of ordinals, so absorption phaenomena are avoided.

‡ Remark that, if the ordered pair \( (a, b) \) is identified with the “Kuratowski doubleton” \( \{\{a\}, \{a, b\}\} \), then, \( \lor \ell(a, b) = \lor \ell (\{\{a\}, \{a, b\}\}) = \ell(a) \lor \ell(b). \) Hence the Cartesian product of \( (E_1, \ell_1) \) and \( (E_2, \ell_2) \) is precisely the labelled set \( (E_1 \times E_2, \ell_1 \lor \ell_2). \)
Nonarchimedean probability

The numerosity theory can have natural application in considering the probability of \textit{infinite} events: one can give the natural combinatorial interpretation as the quotient between the numerosity of the positive cases and that of all possible cases (Of course in the case of elementary events of equal probabilities, but the transfinite sums of the Euclidean field $\mathbb{E}$ allow for dealing in similar way the case of elementary events of arbitrary probabilities)

The Euclidean probability

Let \((\mathbb{U}, \pi)\) be a Euclidean numerosity theory, and let \(U \in \mathbb{U}\) be the space of the events. The Euclidean probability associated to \(\pi\) is \(p : \mathcal{P}(U) \to \mathbb{E}\) such that \(p(E) = \frac{n(E)}{n(U)}\) for all events \(E \subseteq U\).

This probability satisfies the following axioms:

1. \(p(E) \in [0, 1]_{\mathbb{E}}\) for all \(E \subseteq U\)
2. \(p(E) = 0 \iff E = \emptyset\) and \(p(E) = 1 \iff E = U\)
3. \(p(E) + p(H) = p(E \cup H) + p(E \cap H)\)
**conditional probability**

Define as usual the *conditional probability* of the event $E$ given the event $H$ by

$$p(E \mid H) = \frac{p(E \cap H)}{p(H)}.$$

When the probability comes from a Euclidean numerosity, the continuity property of Kolmogorov (or else the $\sigma$-additivity) is replaced and strengthened as follows

4. If $p(E \mid F) \leq p(H \mid F)$ for any finite event $F$, then $p(E) \leq p(H)$. 
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