EUCLIDEAN NUMBERS, EUCLIDEAN NUMEROSITY, AND EUCLIDEAN PROBABILITY

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The field of the Euclidean numbers

From the algebraic point of view, the Eucliean numbers are a *non-Archimedean ordered superfield of the reals* $\mathbb{E}$, with a supplementary structure, the *Euclidean structure*, introduced axiomatically by the operation of (ordinal-indexed) transfinite sum:

$$\sum_{k<\alpha} a_k$$

where the $a_k s$ are *real* numbers, while k and α are ordinals smaller than the first inaccessible ordinal Ω.

We assume that a *transfinite sum* coincides with the *ordinary sum* of the field $\mathbb{E}$ if the number of *non-zero summands* is *finite*.

Let $\quad \mathcal{S}(\Omega, \mathbb{E}) = \left\{\xi \in \mathbb{E}^{\Omega} \,|\, \exists j \in \Omega,\ \forall \beta > j,\ \xi_j = 0\right\}$

be the set of all *eventually* 0 Ω*-sequences* of elements of $\mathbb{E}$, and

let $\quad \Sigma : \mathcal{S}(\Omega, \mathbb{E}) \to \mathbb{E}$ be an $\mathbb{E}$-*linear* map.

For $\xi = \langle \xi_k \mid k \in \Omega \rangle \in \mathcal{S}(\Omega, \mathbb{E})$ define $\quad \Sigma(\xi) := \sum_k \xi_k,$

(the *transfinite sum of the Euclidean numbers* $\xi_k, k \in \Omega$).

We give three natural axioms that rule the behaviour of these transfinite sums.

RA **Real numbers Axiom:**

For all $\xi \in \mathbb{E}$ *there exists* $\mathbf{x} \in \mathcal{S}(\Omega, \mathbb{R}) = \mathcal{S}(\Omega) \cap \mathbb{R}^{\Omega}$ *such that*

$$\Sigma(\mathbf{x}) = \xi$$

Grounding on the axiom RA, in the follwing axioms we restrict ourselves to considering transfinite sums of *real numbers*.

For sake of clarity, we denote general Euclidean numbers by *greek* letters ξ, η, ζ, and real numbers by *latin* letters x, y, z.

DA **Double sum axioms:**

DA1 $\sum_h \sum_k x_{hk} = \sum_k \sum_h x_{hk} = \sum_i \sum_{h \vee k = i} x_{hk}.$

So we may denote any of these double sums by $\sum_{h,k} x_{hk}.$

DA2 *Let* $x_{hk} = 0$ *for* $h, k \geq \omega^j$. *Then*

$$\sum_{h,k} x_{hk} = \sum_i y_i \quad where \quad y_i = \begin{cases} x_{hk} & if \quad i = \omega^j \odot h + k \\ 0 & otherwise \end{cases}.$$

CA **Comparison Axiom:**

For all $\mathbf{x}, \mathbf{y} \in \mathcal{S}(\Omega) \cap \mathbb{R}^{\Omega}$, *if there exists* $\beta \in \Omega$ *such that*

$$\sum_{k \sqsubseteq \alpha} x_k \leq \sum_{k \sqsubseteq \alpha} y_k \quad for\ all \quad \alpha \sqsupseteq \beta, \qquad then \quad \sum_k x_k \leq \sum_k y_k.$$

The *formal inclusion* $\sqsubseteq$ between ordinals is a *lattice partial ordering extending the natural ordering*, such that

$$\{\xi \mid \xi \sqsubseteq \alpha\} \text{ is } \textit{finite} \text{ for all } \alpha \in \Omega$$

It is defined according to the normal forms:*

$$\alpha = \sum_{n=1}^{N} \omega^{j_n} \odot a_n \ \sqsubseteq \ \beta = \sum_{n=1}^{N} \omega^{j_n} \odot b_n \iff a_n \leqq b_n \text{ for } 1 \leq n \leq N.$$

So all sums $\sum_{k \sqsubseteq \alpha}$ are *ordinary finite sums of real numbers*.)

* The name formal inclusion should recall that the respective coefficient free normal forms, considered as *multisets*, are indeed contained one inside of the other one.

These simple and natural axioms endow $\mathbb{E}$ with a very rich structure. First a few simple consequences.

- **Product formula**:

$$(\sum_h x_h)(\sum_k y_k) = \sum_i \sum_{h \vee k = i} x_h y_k.$$

- **Translation invariance:** Let $x_k = 0$ for $k \geq \omega^j$. Then

$$\sum_k x_k = \sum_i y_i, \quad \text{where} \quad y_i = \begin{cases} x_k & if \quad i = \omega^j \odot h + k \\ 0 & otherwise \end{cases}.$$

- **Associativity**:

  Put $\quad \sum_{k \in [\alpha,\beta)} \xi_k = \sum_k \xi_k \left(\chi_\beta(k) - \chi_\alpha(k)\right)$. Then

$$\sum_{k \in [0,\omega^j \odot \beta)} x_k = \sum_{h \in [0,\beta)} \sum_{k \in [\omega^j \odot h, \omega^j \odot (h+1))} x_k.$$

The theory of the Euclidean numbers combines the Cantorian theory of ordinal numbers with Non Standard Analysis (NSA).

1. Every Euclidean number can be obtained as a transfinite sum of real numbers; a transfinite sum of Euclidean numbers is well defined, and can be obtained as *limit of ordinal-indexed (finite) partial sums*, when suitable topologies are given to $\mathbb{E}$ and to the ordinals.

2. Any accessible ordinal $\alpha \in \Omega$ can be identified with the transfinite sum of α ones in $\mathbb{E}$; this identification is consistent with the *natural ordinal operations* $\oplus$ *and* $\odot$, so $\mathbb{E}$ is a sort of *natural* extension of the *semiring* of the ordinal numbers in Ω.

3. The Euclidean numbers are a *hyperreal field*; more precisely $\mathbb{E}$ is the *unique saturated real closed field having the cardinality of* Ω. It is isomorphic to the hyperreal Keisler field [keisler76], and every ordered field of accessible cardinality is (isomorphic to) a subfield of $\mathbb{E}$.

   The ordinals as embedded in $\mathbb{E}$ are a (proper) subsemiring of the hypernatural numbers $^*\mathbb{N}$ of $\mathbb{E}$.

4. the Euclidean numbers are strictly related to the notion of *numerosity*, introduced by Benci, Di Nasso and Forti in order to save the five Euclidean common notions on *magnitudines*. In fact, $\mathbb{E}$ can be charachterized as the hyperreal field generated by the real numbers and the semiring of numerosities

We have chosen to call $\mathbb{E}$ the field of the *Euclidean* numbers for two main reasons: firstly, this field arises in a numerosity theory (including all subsets of Ω), which saves all the Euclidean common notions, including the fifth

The whole is greater than the part,

in contrast to the Cantorian theory of cardinal numbers.

Secondly, the field $\mathbb{E}$ describes the Euclidean continuum better than the *real field* $\mathbb{R}$, at least when considering the *set theoretic reduction* of $\mathbb{R}$. This last point has been extensively dealt with in [Benci-Freg.]. As a simple example, note that, contrary to the Euclidean requirement, a segment cannot be divided by a point into two congruent pieces.

The notion of numerosity

A measure of size for arbitrary sets should abide by the famous five common notions of Euclids Elements, which traditionally embody the properties of any kind of magnitudes,

1. *Things equal to the same thing are also equal to one another.*
2. *And if equals be added to equals, the wholes are equal.*
3. *And if equals be subtracted from equals, the remainders are equal.*
4. *Things* [exactly] *applying onto one another are equal to one another.*
5. *The whole is greater than the part.*

NB Translating $\epsilon\phi\alpha\rho\mu o\zeta o\nu\tau\alpha$ by *"applying* [exactly] *onto"*, instead of the usual *"coinciding with"* seems to give a more appropriate rendering of the Euclidean usage of the verb $\epsilon\phi\alpha\rho\mu o\zeta\epsilon\iota\nu$, which refers to superposition of congruent figures.

see Euclid, *The Elements* (T.L. Heath translator), Dover, New York 1956.

How to save the Euclidean common notions?

It is worth noticing that taditional geometry satisfies the Euclidean common notions because there is a restricted class of "exact applications" (*e.g.*, the *rigid equidecompositions* of polygons). So the question arises as to *which correspondences* can be taken as "exact applications" in order to fulfill the five Euclidean common notions.

Cantor himself, besides cardinals based on general bijections, introduced *ordinal* numbers, assigned to sets endowed with a *wellordering*, restricting the "exact applications" to the order preserving bijections. But the Euclid's principle EP still badly fails.

The *Euclidean* numbers are an *extension* of the ordinals, suitable to provide a notion of size satisfying *all* the Euclidean common notions for an appropriate class of *"labelled sets"*.

A labelled set can be viewed as a generalization of a wellordered set, because the latter can be naturally labelled by the unique order isomorphism with an (initial segment of an) ordinal. In fact it will turn out that any labelled set is "equinumerous" to a *set of ordinals*.

The idea is that by putting an appropriate *labelling* on arbitrary sets, the *label preserving bijections* (intended as "exact applications") might produce the "nonnegative integers" of the Euclidean numbers (whence their name).

Labelled sets

A *labelled set* is a pair (E, ℓ), where

- E is a set of cardinality less than the first inaccessible Ω;
- $\ell : E \to \Omega$ is a function (the *labelling function*) such that
  1. the set $\ell^{-1}(x)$ is finite for all $x \in \Omega$;
  2. $\ell(x) = x$ for all $x \in E \cap \Omega$.
- Two labelled sets (E_1, ℓ_1) and (E_2, ℓ_2) are *isomorphic* if there is a biunique map $\phi : E_1 \to E_2$ such that
$$\ell_2(\phi(x)) = \ell_1(x) \quad \text{for all } x \in E_1.$$
- Two labelled sets (E_1, ℓ_1) and (E_2, ℓ_2) are *coherent* if
$$\ell_1(x) = \ell_2(x) \quad \text{for all } x \in E_1 \cap E_2.$$

- The *basic operations* on labelled sets are the following:

1. Subset - A *subset* of a labelled set (E, ℓ) is a labelled set
$$\left(F, \ell_{|F}\right) \quad \text{where} \quad F \subseteq E;$$
2. Union - The *union* of two *coherent* labelled sets (E_1, ℓ_1), (E_2, ℓ_2) is the labelled set
$$(E_1 \cup E_2, \ell) \quad \text{where} \quad \ell(x) = \begin{cases} \ell_1(x) & if\ x \in E_1 \\ \ell_2(x) & if\ x \in E_2 \end{cases}$$
3. Cartesian product - The *Cartesian product* of two labelled sets (E_1, ℓ_1), (E_2, ℓ_2) is the labelled set
$$(E_1 \times E_2, \ell) \quad \text{where} \quad \ell(x_1, x_2) = \ell_1(x_1) \vee \ell_2(x_2)$$

- A family $(\mathcal{A}, \ell)$ of *pairwise coherent* labelled sets is *closed* if it is closed under the three basic operations.

The Euclidean numerosity

A *Euclidean numerosity theory* is a pair $(\mathbb{U}, \mathfrak{n})$, where

- $\mathbb{U}$ is a *closed family* of labelled set, and
- $\mathfrak{n} : \mathbb{U} \to \mathbb{E}$ is the *Euclidean numerosity function* associating to each labelled set $(E, \ell) \in \mathbb{U}$ its euclidean numerosity
$$\mathfrak{n}(E, \ell) = \sum_k |\ell^{-1}(k)| \in \mathbb{E};$$
- The set of numerosities of $(\mathbb{U}, \mathfrak{n})$ is the range $\mathfrak{N} = \mathfrak{n}[\mathbb{U}] \subseteq \mathbb{E})$.

Remark Given a coherent family $(\mathcal{A}, \ell)$ of labelled sets, there exists a least closed family $\mathbb{U}(\mathcal{A}, \ell)$ including $\mathcal{A}$, called the *closure of* $(\mathcal{A}, \ell)$. We omit the labelling function ℓ when it is clear from the context.

Canonical examples

Recall that we identify the natural numbers with the finite ordinal numbers, and the accessible ordinals with the corresponding Euclidean numbers.

- Let $F \subset \Omega$ be *finite*. Then $\mathbb{U}(\{F\})$ contains only finite sets, so $\mathfrak{n}$ is the *finite cardinality*, and $\mathfrak{N} = \mathfrak{n}[\mathbb{U}(\{F\})] = \mathbb{N} \subset \mathbb{E}$.

- $(\mathbb{U}[\mathbb{N}], \mathfrak{n})$ is the "simplest" numerosity theory containing *infinite* sets, and $\mathfrak{N} = \mathfrak{n}[\mathbb{U}(\mathbb{N})] \subset {}^*\mathbb{N} \subset \mathbb{E}$..

- The *canonical numerosity theory* $(\mathbb{U}(\Omega), \mathfrak{n})$ is the "simplest" theory which contains all the (accessible) ordinal numbers, and $\Omega \subseteq \mathfrak{N} = \mathfrak{n}[\mathbb{U}(\Omega)] \subseteq {}^*\mathbb{N} \subset \mathbb{E}$.

Main properties of the Euclidean numerosities

Theorem Let $(\mathbb{U}, \mathfrak{n})$ be a Euclidean numerosity theory. Then one has, for all $A, B \in \mathbb{U}$:

Sum-Difference: $\mathfrak{n}(A \cup B) = \mathfrak{n}(A) + \mathfrak{n}(B) - \mathfrak{n}(A \cap B)$;

Part-Whole: $A \subset B \Longrightarrow \mathfrak{n}(A) < \mathfrak{n}(B)$;

Cartesian Product: $\mathfrak{n}(A \times B) = \mathfrak{n}(A) \cdot \mathfrak{n}(B)$;

Comparison: if the 1-to-1 map $T : A \to B$ preserves labels, then $\mathfrak{n}(A) \leq \mathfrak{n}(B)$.

Moreover, if $\Omega \subseteq \mathcal{A}$, then

- each (accessible Von Neumann) ordinal is its own numerosity,
- each set in $\mathbb{U}$ is equinumerous to a set of ordinals.

Saving the five Euclidean common notions

The Euclidean numerosities satisfy *all* the five Euclidean common notions, when interpreted in the natural way:

1. Two labelled sets are *equal* (in size) if they have the same numerosity;

2. The *addition* of two (disjoint) labelled sets is given by their union;

3. The *subtraction* of a labelled set from a (coherent) superset is given by relative complement;

4. Two labelled sets *(exactly) apply onto* one another if they are isomorphic;

5. A *part* of a labelled set is just a (coherent) subset.

Other properties of the Euclidean numerosities

Translation invariance: $\forall E \subseteq \omega^j,\ \mathfrak{n}\left(\{\omega^j + \xi \mid \xi \in E\}\right) = \mathfrak{n}(E)$;

Homothety invariance: $\forall E \subseteq \omega^{\omega j},\ \mathfrak{n}\left(\omega^{\omega j}\xi \mid \xi \in E\}\right) = \mathfrak{n}(E)$;

Cartesian product: $\forall \alpha, \beta \in \Omega,\ \mathfrak{n}(\alpha \times \beta) = \alpha\beta;^{\dagger}$

Label the set $\mathcal{P}_\omega(E)$ of the finite parts of the labelled set (E, ℓ) by the label $\vee\ell$ such that $\vee\ell\left(\{a_1, ..., a_n\}\right) = \bigvee_{k=1}^{n} \ell(a_k)$.[‡]

If the family $\mathbb{U}$ is closed under the operation $\mathcal{P}_\omega$, one has

Finite parts: $\mathfrak{n}(\mathcal{P}_\omega(E)) = 2^{\mathfrak{n}(E)}$.

[†] Notice that the product of the field $\mathbb{E}$ agrees with the *natural* product of ordinals, so *absorption phaenomena* are avoided.

[‡] Remark that, if the *ordered pair* (a, b) is identified with the "Kuratowski doubleton" $\{\{a\}, \{a, b\}\}$, then, $\vee\ell(a, b) = \vee\ell\left(\{\{a\}, \{a, b\}\}\right) = \ell(a) \vee \ell(b)$.. Hence the Cartesian product of (E_1, ℓ_1) and (E_2, ℓ_2) is precisely the labelled set $(E_1 \times E_2, \ell_1 \vee \ell_2)$.

Nonarchimedean probability

The numerosity theory can have natural application in considering the probability of *infinite* events: one can give the natural combinatory interpretation as the quotient between the numerosity of the positive cases and that of all possible cases (Of course in the case of elementary events of equal probabilities, but the transfinite sums of the Euclidean field $\mathbb{E}$ allow for dealing in similar way the case of elementary events of arbitrary probabilities)

see V. Benci, H. Horsten, S. Wenmackers - Non-Archimedean probability. *Milan J. Math.* **81** (2013), 121–151.

The Euclidean probability

Let $(\mathbb{U}, \mathfrak{n})$ be a Euclidean numerosity theory, and let $U \in \mathbb{U}$ be the space of the events. The Euclidean probability associated to $\mathfrak{n}$ is $p : \mathcal{P}(U) \to \mathbb{E}$ such that $p(E) = \frac{\mathfrak{n}(E)}{\mathfrak{n}(U)}$ for all events $E \subseteq U$.

This probability satisfies the following axioms:

1. $p(E) \in [0, 1]_{\mathbb{E}}$ for all $E \subseteq U$
2. $p(E) = 0 \iff E = \emptyset$ and $p(E) = 1 \iff E = U$
3. $p(E) + p(H) = p(E \cup H) + p(E \cap H)$

conditional probability

Define as usual the conditional probability of the event E given the event H by $p(E \mid H) = \frac{p(E \cap H)}{p(H)}$.

When the probability comes from a Euclidean numerosity, the continuity property of Kolmogorov (or else the σ-additivity) is replaced and strengthened as follows

4. If $p(E \,|\, F) \leq p(H \,|\, F)$ for any finite event F, then $p(E) \leq p(H)$.

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