L.E.J. Brouwer and A. Heyting on foundational labels: their creation and use

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The use of the three labels (logicism, formalism, intuitionism) to denote the three foundational schools of the early twentieth century is now firmly part of the literature. Still, they were not introduced by the founding fathers each for his own school. Furthermore, neither their number nor their adoption has been stable over the twentieth century. In this talk we will see the role and attitude that intuitionists of the first era (Brouwer and Heyting) have had in the production and use of foundational labels.

I will advance the thesis that not only the creation but also the use of labels, far from being a mere gesture of academic reference to literature, can be a sign of the cultural operation each scholar wants to do.

Prof. dr. L. E. J. Brouwer,
faculteit der wis- en natuurkunde
(wiskunde, mechanica).

Brouwer’s thesis (1907)

In his doctoral thesis Brouwer used the labels at hand at the moment on the foundational “market” (i.e. those mentioned in the mathematical/philosophical journals where the papers by Cantor, Poincaré, Hilbert, Couturat, Russell had appeared). Therefore, we find: “axiomatics”, “cantorianism” and “logistics” (as proposed by Couturat). We do not find either “logicism”, because such foundational label appeared only in 1928, or “intuitionism”, because Brouwer had not yet chosen anything specific for his own viewpoint.

Brouwer analyzed the various foundational approaches starting from his own: his own was the right one and the differences with it explained the failure of other approaches.
On this basis, he began to analyse : 1) the foundations on axioms, 2) Cantor’s transfinite numbers, 3) the Peano-Russell logistic, 4) the logical foundation after Hilbert.

The first field embraced the “recent investigations of Pasch, Schur, Hilbert, Peano, Pieri”, that had pointed out some “holes” in Euclidean treatment of geometry: they have “convincingly demonstrated that [Euclidean geometry] as a logical linguistic structure is imperfect, namely that here and there tacitly axioms are introduced”. However, Brouwer stigmatized that their very target were not Euclidean imperfections, but pathological geometries. “They constructed linguistic structures and then they required a proof of consistency, but nobody proved that consistency was a condition sufficient for existing mathematically.“

The second field that Brouwer considered was Cantorianism, that he criticized because “Cantor loses contact with the firm ground of mathematics” (CW I, 81) in his definition of the second number class. Namely, the concept of “den Inbegriff aller’ mentioned something which cannot be thought of, for a totality constructed by means of ‘and so on’ could only be thought of if ‘and so on’ referred to an ordertype ω of equal objects, and this was not the case.

Then he passed to Peano and Russell’s treatment of logic. Brouwer saw it as an attempt to put a remedy to the fact that classical logic was inadequate for mathematics. “The logisticians consider the propositional functions as the free origin of logic and mathematics […] But in the intellect one cannot give a linguistic system of statements and propositional functions priority over mathematics, for no assertions about the external world can be intelligently made besides those that presuppose a mathematical system that has been projected on the external world” (CW I, p. 88). Therefore it was not surprising that they, like the Cantorians, came up against contradictions.

The conclusion that he drew about logistics was that it was not suitable as a foundation of mathematics because it was separated from mathematics and the best to which it could aim was being a faithful stenographic copy of the language of mathematics (“which itself is not mathematics but no more than a defective expedient for men to communicate mathematics to each other and to aid their memory for mathematics” - CW I, p. 92).

Thereafter he turned back to Hilbert, that gave "the most uncompromising conclusion of the methods we attack, which illustrates most lucidly their inadequacy" (CW I, p. 92).

He stressed that Hilbert was even more criticizable than "the logicians" because in his works it was possible to find a list of stages that were confused:

1. Construction of intuitive mathematical systems
2. Mathematical speaking or writing (the expression of 1)
3. The mathematical study of language: “we notice logical linguistic structures, raised according to principles from ordinary logic or through the logic of relations”
4. Forgetting the sense of the elements of the logical figures in 3. and imitating the construction of these figures by a new mathematical system of second order.
5. The language that may accompany 4.
6. The mathematical study of language 5.
7. Forgetting the sense of the elements of the logical figures in 6. and imitating the construction of these figures by a new mathematical system of third order.
8. The language that may accompany 5.

1911

- It was only in the 1911 review of the Mannoury volume "Methodologisches Philosophisches und zur Elementar-Mathematik" that Brouwer suddenly introduced the contraposition intuitionism vs formalism (that was not present in Mannoury’s book): “[…] the author defends the ‘formalist’ conception, which has also been advocated by Dedekind, Peano, Russell, Hilbert and Zermelo, against ‘intuitionists’ like for instance Poincaré and Borel.

This formalist conception recognizes no other mathematics than the mathematical language and it considers it essential to draw up definitions and axioms and to deduce from these other propositions by means of logical principles which are also explicitly formulated beforehand” (CW I, 121).

After defining formalism, Brouwer wondered what could be the reason for accepting those axioms and recalled that Russell’s answer was to verify the logical existence of mathematical entities, while Hilbert’s answer had been the project to verify that the logical figure of ‘contradiction’ could not be derived by the axioms. He ended by stressing that both Russell and Hilbert could not do without “the intuitive application of complete induction” and therefore “they have invigorated by their reasoning rather intuitionism than formalism” (CW I, 121).

Furthermore, he added that Mannoury could criticise intuitionism because he had in mind only Poincaré’s version of it, that presented two weak points: the rejection of every infinite number, including the denumerable, and the identification of mathematical existence with non-contradictoriness. Brouwer was sure that “it is only after these mistakes have been redressed, and after the basic intuition of two-ity has been accepted, that intuitionism becomes invulnerable” (p. 122).

In Mannoury's book were described two opposite viewpoints about mathematics: Kantianism and Symbolism. Thus, the question arises how Brouwer came to employ the labels formalism/intuitionism. Mannoury presented the contraposition Kantianism/Symbolism in the index of the book (p. 262), inside the third group of subjects belonging to the first chapter (of the second part of the book) devoted to mathematical logic: "Kritik des Symbolischen Logik; die Beurteilung der Widerspruchslosigkeit der logischen Formeln; Kantianismus und Symbolismus (139-149)". In the inner pages of the book we find "Kantianismus" as attached mainly to Poincaré, defined as "the talented representant of Kantianism in mathematics (p. 144).

The authors quoted on the opposite side are Peano, Couturat and Hilbert. Mannoury specifies that among other (unnamed) Kantianists there is Aurel Voss, that, in his 1908 lecture “Über das Wesen der Mathematik”, defended “the higher meaning of mathematics, in opposition to *formalism*”. Therefore, the word “Formalism” for the enemies of “Kantianism” is present in a footnote.

We can imagine that Brouwer decided to change the label “Kantianism” for his foundational school in order to point out the novelty of his own position (even if he admitted his ‘debt’ to Kant). Since the intuition of two-ity had been indicated by himself as the key for granting the unvulnerability of his position, it seems comprehensible that he used the label “intuitionism”. Once he decided to change the main of the two labels, it could have come natural to him to change also the other, that was more “unstable” inside the book and that was already substituted by “formalism” in a footnote.

It should be considered also that Felix Klein, in his first Evanston lecture (1893) had distinguished three main categories among mathematicians - logicians, formalists and intuitionists: “the word logician is here used, of course, without reference to the mathematical logic of Boole, Peirce, etc; it is only intended to indicate that the main strength of the men belonging to this class lies in their logical and critical power, in their ability to give strict definitions, and to derive rigid deductions therefrom” (p. 2). As an example he quoted Weierstrass.

“The formalists excel mainly in the skillful formal treatment of a given question, in devising for it an “algorithm”. “To the intuitionists, finally, belong those who lay particular stress on geometrical intuition (Anschauung), not in pure geometry only, but in all branches of mathematics. What Benjamin Peirce has called ‘geometrizing a mathematical question’ seems to express the same idea”. Examples: Lord Kelvin and von Staudt.

We see that the content of the categories is different from Brouwer’s meaning, but the labels “formalism” and “Intuitionism” had been coined. Brouwer could have been inspired by them

1912

In 1912, in his introductory lecture “Intuitionism and formalism”, he stated that there were two main points of view on what grounded the exactness of mathematics: “The question where mathematical exactness does exist, is answered differently by the two sides; the intuitionist says: in the human intellect; the formalist says: on paper” (CW I, p. 125).

He traced back an *old form of intuitionism* in Kant, but added that Kant's intuitionism was weak. It became more credible when he abandoned apriority of space and built mathematics on the only intuition of time. "In the construction of these sets neither the ordinary language nor any symbolic language can have any other role than that of serving as a non-mathematical auxiliary, to assist the mathematical memory or to enable different individual to build up the same set." (CW I, p. 128).

Brouwer passed to describe the formalists as scholars that started from the belief that human reason did not have at its disposal exact images either of straight lines or of large numbers (numbers larger than three, for example). Then, they concluded that such entities "do not have existence in our conception of nature any more than in nature itself", but they grounded their non-mathematical conviction of legitimacy of their systems on the efficacy of their projection into nature.

“For the formalist, therefore, mathematical exactness consists merely in the method of developing the series of relations […] And for the consistent formalist these meaningless series of relations to which mathematics are reduced have mathematical existence only when they have been represented in spoken or written language together with the mathematical-logical laws upon which their development depends, thus forming what is called symbolic logic” (CW I, p.125). In order to be sure of the consistency of the language that they used, formalists avoided daily language and introduced new ones. Here we find Peano labelled as formalist.

According to Brouwer, intuitionists and formalists agreed as for finite sets: in that field the two tendencies differed solely in their method, not in their results. On the contrary, when infinite sets are considered, “the formalist introduces various concepts, entirely meaningless to the intuitionist, such as ‘the set whose elements are the points of space’, ‘the set whose elements are the continuous functions of a variable’, etc.” (CW I, p. 130) Brouwer ended his lecture by stating that he saw no hope that an agreement in a finite period could be reached. He quoted Poincaré: “Les hommes ne s’entendent pas, parce qu’ils ne parlent pas le même langue et qu’il y a des langues qui ne s’apprennent pas” [The men disagree, because they do not speak the same language and there are languages that cannot be learned] (CW I, p. 138).

1928

In his 1928 “Intuitionistische Betrachtungen über den Formalismus” Brouwer went on with his „duel“ with formalism and listed four viewpoints that the intuitionists asserted and that he was sure that, soon or later, also formalists would share: there would be the end of the “Grundlagenstreit” and from that time on, the choice between formalism and intuitionism would be only a question of taste.

The four viewpoints mentioned were:

1) The formalists aim to build a formal image of mathematics while they also have an intuitive theory of the laws of such a construction. They admit that intuitionistic mathematics of the integers is indispensable for their intuitive theory.

2) The refusal of a blind application of the principle of excluded middle, that has a sure validity only for finite domains.

3) The identification of the principle of excluded middle with the principle of the solvability of every mathematical problem.

4) The knowledge that justifying formalistic mathematics through the proof of its non-contradictoriness contains a vicious circle: it is grounded on the law that let to pass from the double negation to the affirmation; but this law is grounded, at its turn, on the principle of excluded middle.

The first viewpoint, “prepared by Poincaré”, was expressed by Brouwer in 1907 [CW I, 94] and shared in formalistic literature as testified by the use of the word “metamathematics”. The second viewpoint, that appeared in Brouwer 1908 for the first time, had somehow been accepted by the formalists insofar Hilbert admits “die beschränkte inhaltliche Gültigkeit“ of the principle of excluded middle. Still, Brouwer noted, he himself extended his doubt to all of Aristotelian logical laws.

The third and the fourth viewpoints had not yet been accepted by the formalists.

1928 is also the year of Brouwer’s “Berliner Gastvorlesungen”. Van Dalen published in 1991 Brouwer’s manuscripts of the lessons [Brouwer 1991]. We see the same structure that we will read in the postwar years. In the first chapter devoted to the historical position of intuitionism, he distinguished three periods:

1) The first lasted till the 19th century, was characterized by the belief in the existence of properties of time and space, independent both of language and logic, and was called by Brouwer ”Kantian viewpoint”. The period finished as a consequence of the non-euclidean geometry and of the theory of relativity that interrupted the belief in the Kantian theory of space and let mathematics be based on the theory of numbers.

2) Therefore, the second period was characterized by the arithmetization of geometry [*old-formalist school* (Dedekind, Peano, Russell, Couturat, Hilbert, Zermelo)]. In the meanwhile, the *pre-intuitionistic school* (Kronecker, Poincaré, Borel, Lebesgue) was “completely different”. Still, pre-intuitionism went on with using logic (including the principle of excluding middle) in mathematical inferences (Herleitungen). To the same period belongs the *new-formalistic school*, represented by Hilbert, Bernays, Ackermann and von Neumann.

3) The third period was characterized by the two acts (Handlungen) of *Intuitionism*.

Through the first act, it separated mathematics and logic, and grounded mathematics on the languageless activity of mind having its origin in the perception of time. Through the second act it recognized the possibility of producing: 1) the infinitely proceeding sequences of mathematical entities (previously acquired); 2) mathematical species, i.e. properties supposable for mathematical entities previously acquired .

We see that Brouwer kept the two labels that he had introduced (formalism vs intuitionism), by adding a Kantian viewpoint that in the 1912 lecture simply belonged to “intuitionism” and that in his final works will disappear at all. Furthermore, he distinguished inside each label two standpoints: an old (or pre-) one and a new one. It is clear that in this way Brouwer could stress that his intuitionism was the end point of a finalized path, and not a variant of an already present school.

In “Mathematik, Wissenschaft und Sprache” (talk given in Vienna in 1928), Brouwer mentioned the formalistic school (CW I p. 422) and stated that its fault resided in its belief in classical logic. The origin of such belief consisted in the fact that their laws were trustable when they were referred to finite domains. But this trustworthiness led man to a superstitious faith in the miraculous power of language. *Brouwer opposed again the intuitionists to the formalists*: the intuitionists destroyed the confidence in logical laws when applied to infinite domains by giving counterexamples to the validity of the principle of excluded middle in those domains and criticized the formalists for building a linguistic castle instead of a mathematical building.

“Die Struktur des Kontinuums” (second Vienna talk) was devoted to explain the novelty of Brouwer’s treatment of the continuum through the exploitation of the duo-unity in terms of lawless sequences. In order to do this, Brouwer pointed out the different approaches to the continuum in recent history of mathematics. The partition among schools was expressed as follows: his position was called “*intuitionist*”; then there were the *formalists* (stemming from Dedekind, Peano, Russell, Zermelo and Hilbert) which paid attention only to mathematical language and took care of avoiding from the theorization of the continuum the production of contradictions; and the *old-intuitionists* (stemming from Poincaré and Borel) for whom only the denumerable part of the continuum had a content, i.e. could be built by constructive means starting from the intuition of the duo-unity, while for the more-then-denumerable continuum the reference to a merely linguistic source was necessary.

During the 1928 Brouwer had a hard conflict with Hilbert, described in detail in van Dalen’s 1990 article “The battle of the frog and the mice”. Hilbert re-founded the “Mathematische Annalen” to strike Brouwer off the editorial committee. Although the other editors tried to bake the pill, Brouwer felt very hurt. Consequently, we find a break of his publications, with some small exceptions, from 1930 to 1948.

Logicism at the horizon

In 1928, in his revised edition of «Einleitung in die Mengenlehre», Abraham Fraenkel distinguished three Begründungsarten of Set Theory: logicism, intuitionism and formalism, by specifying that he preferred logicism.

His preference for the logicism in the sense of "insertion of mathematics into the wider framework of logic" was not only due to the hope that the difficulties linked to the reducibility axiom would be solved, but even if it would not happen, insofar as also the other schools were not successful, he would count this as a proof of the limits of human thought regarding such questions

Fraenkel distinguished between “logistics” and “logicism”. He stressed (1928, 263) that the first label was not univocal, because it was used in the literature both to indicate the formal aspects of the new logic that had been developed in the 19th century (algebra of logic, symbolic logic, etc.) and for the foundational point of view. Therefore, he believed that it was better to indicate the foundational school through a specific label: “Logizismus”

In 1929, in his «Abriss der Logistik mit besonderem Rücksichtigung des Relationstheorie und ihre Anwendungen», Carnap presented “Logistik oder symbolische Logik” by stressing that the whole mathematics was a branch of it. In a footnote [p. 2] he listed Ziehen, Meinong and the three scholars of the Geneva 1904 symposium (Couturat, Itelson, Lalande) as those who had proposed the label. Then he specified that: “A philosophical tendency with a strong or, perhaps, excessive emphasis on the logical point of view is not called 'logistisch', but better, as sometimes already customary, ‘logizistisch’, ‘Logizismus’. (p.3) Therefore, in 1929, he seemed to use ‘Logizismus’ more in a general philosophical sense than with a mathematical-foundational meaning.

Only in 1931 Carnap presented, in the proceedings of the Königsberg Colloquium, published in the second volume of “Erkenntnis”, three different foundational schools that he labelled resp. as logicism, intuitionism and formalism. The aim of the congress was to make a budget of their situation at the beginning of the Thirties. The school that he represented was logicism and he pointed out the relationship between mathematics and logic as one of the more intriguing questions around the problem of the foundations. He stated that logicism saw mathematics as a part of logic so that mathematical concepts were deducible from logical concepts and mathematical theorems were provable starting from logical asserts.

Carnap saw the very problem of logicism in the will of avoiding both the axiom of reducibility and the division of real numbers into different orders.

Then, he looked also for similarities between the various schools. The similarity between logicism and intuitionism consisted in the fact that for both of them no concept could be built on the only axiomatic basis. Concepts had to be constructed out of undefined basic properties of a given domain according to some rules of constructions in a finite number of steps. The similarity between logicists and formalists consisted in the fact that during the deduction in both cases no reference to the meaning of the words was considered.

Brouwer: the final solution

We have seen that in Brouwer’s Berliner lectures (1928) he had introduced a difference between old formalism and new formalism. One could expect that after 1931 he used Gödel’s results as a weapon against formalists, but he did not. On this purpose, Hao Wang reported: “In the spring of 1961 I visited Brouwer at his home. He discoursed widely on many subjects. Among other things he said that he did not think Gödel 's incompleteness results are as important as Heyting's formalization of intuitionistic reasoning, because to him Gödel‘s results are obvious (obviously true)”.

Van Atten (2009) observed that Brouwer's reaction with respect to the first incompleteness theorem was readily understandable. Indeed, it had been an argument of Brouwer's that had stimulated Gödel in finding the first theorem, as reported in Carnap’s diary note on December 12, 1929. According to van Atten, the second incompleteness theorem, on the other hand, must have surprised Brouwer, given his “optimism” in the 1920s about the formalist school achieving its aim of proving the consistency of formalized classical mathematics, testified by the following quote that appeared in a paper rich of counterexamples to the principle of excluded middle, where he described formalism as a reaction to the antinomies that had appeared in mathematics due to the blind use of classical logic (in particular that of excluded middle).

“We need by no means despair of reaching this goal [of a consistency proof for formalized mathematics], but nothing of mathematical value will thus be gained: an incorrect theory, even if it cannot be inhibited by any contradiction that would refute it, is none the less incorrect, just as a criminal policy is none the less criminal even if it cannot be inhibited by any court that would curb it.” [Brouwer 1924, 3, CW I, 270]

Brouwer alluded to Gödel’s results both in his *Cambridge Lectures* and in his 1952 paper, by stating : “The hope originally fostered by the Old Formalists that mathematical science erected according to their principles would be crowned one day with a proof of non-contradictority, was never fulfilled, and, nowadays, in view of the results of certain investigations of the last few decades, has, I think, been relinquished”. (Brouwer 1952B, 508)

Still, Brouwer did not stress the surprising result in order to support his position. I suppose that, since he had previously attached no value to the achievement of a consistency proof, the impossibility of achieving it however did not appear to him a relevant point in his argumentation against formalism: the impossibility of achieving something useless must have seemed to him not particularly interesting. Furthermore, he considered the search for non-contradictoriness of mathematical science typical of an old-formalism, that had been however “shaken” by pre-intuitionists.

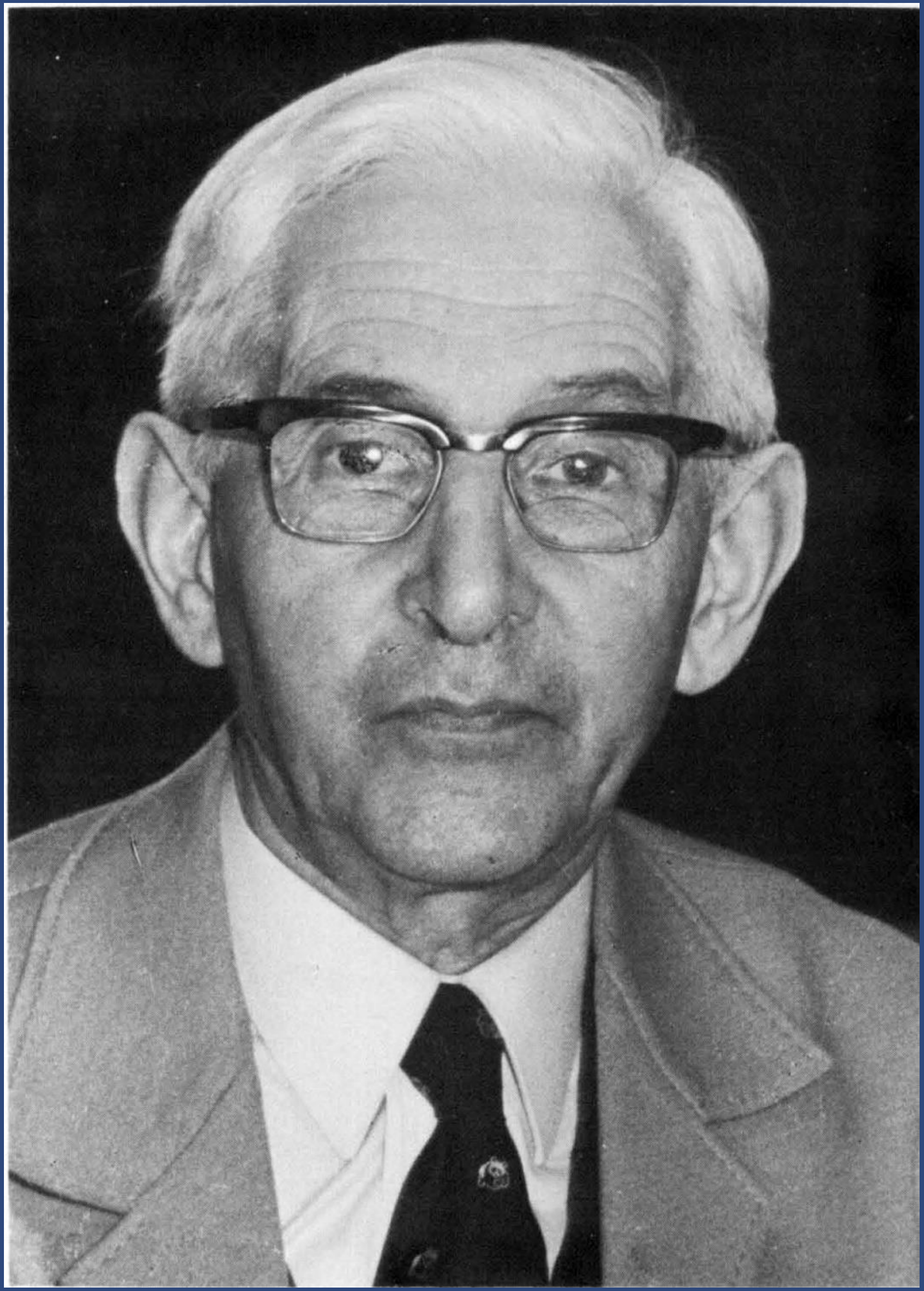
He presented again an evolution from an old period (called "observational period" – instead of "Kantian period"), he asserted that the new formalist school, founded by Hilbert, came out when the old formalist standpoint "had been badly shaken" by pre-intuitionist criticism, that had stressed the essential difference between logic and mathematics and an autonomy of the so-called separable parts of mathematics from. In his new formalism, Hilbert made use of the intuition of natural numbers and of complete induction, and postulated existence and exactness independent of language only for meta-mathematics. Brouwer posed the intervention of (his) intuitionism here, after Hilbert's new-formalism, and depicted a situation "after" Hilbert's viewpoint in this way:

“The situation left by Formalism and Pre-intuitionism can be summarized as follows: for the elementary theory of natural numbers, the principle of complete induction, and more or less considerable parts of algebra and theory of numbers, exact existence, absolute reliability, and non-contradictory were universally acknowledged, independently of language and without proof. There was little concern over the existence of the continuum. Introduction of a set of pre-determinate real numbers with a positive measure was attempted by logico-linguistic means, but a proof of the non-contradictory existence of such a set was lacking. For the whole of mathematics the rules of classical logic were accepted as reliable aids in the search for exact truths” (CW I, p. 509). In this situation, his intuitionism intervened through the famous two acts. Then, Brouwer described the main results of his theory of the continuum (for instance, his fan theorem), and he did not add anything more about the other schools.

We have seen that Brouwer in his doctoral thesis (1907) presented the points of view of “cantorians”, “axiomaticians” and “logistics” and confronted his perspective (not yet labelled) with each of them. From the 1911 review of the Mannoury volume "Methodologisches Philosophisches und zur Elementar-Mathematik" on, Brouwer presented the foundational scene of the last century divided into two main blocks: the "formalists" (including Dedekind, Peano, Russell, Hilbert and Zermelo) and the "intuitionists" (represented by Poincaré and Borel).

The following year, Brouwer chose the label "neo-intuitionism" for his foundational school, and kept the label "formalism" to indicate all the others. From the 1930 conference of Königsberg on (attended by Carnap, Heyting and von Neumann), the label “formalism” was attached to the Hilbert school, while the label "logicism", which appeared (in the foundational context) two years before in the revised edition of the book *Einleitung in die Mengenlehre* by A. Fraenkel, was attributed to the school of Frege-Russell (represented at the conference by Carnap).

Brouwer didn’t share such a tripartition, and in his later writings he settled on the contrast Intuitionism-formalism, as given in his 1928 “Berliner Lectures”, where he had distinguished inside the two mainstreams old and new schools: old formalists were Dedekind and Zermelo, new formalist was Hilbert; pre-intuitionists were Poincaré and Lebesgue, intuitionist was Brouwer himself. I propose the explanation that Brouwer, as a founding father that wanted to present himself as new and definitive, used two labels in order to describe the entire landscape as it would be divided into "self" and "non-self": it was a way for him to focus on his own perspective and to reaffirm its character of absoluteness.



Arend Heyting

Arend Heyting was Brouwer’s student and took part to the Königsberg conference in 1930. In his 1934 he published “Mathematische Grundlagenforschung. Intuitionismus. Beweistheorie”.

The plan of the book that Heyting proposed in his first letter was the following: “1. Short historical introduction. (Poincaré); 2. The paradoxes; attempts at resolution apart from the three principal directions; 3. The calculus of logic; its further development (Americans); logicism; 4. Intuitionismus; 5. Formalism; 6 Other standpoints; 7. Relations between the different directions; 8. Mathematical and natural science.” Gödel should have covered the first three chapters, but he tried to re-adjust the distribution. In a draft of a letter (VIII 1931) he explained that metamathematics was hardly separable from logistic (Logistik) because metamathematics was a theory of the linguistic forms: so he suggested to keep himself that part, by leaving to Heyting the formalization of intuitionism (in addition to intuitionism and semi-intuitionism).

Later (IX 1931), he proposed again to treat he himself metamathematics as an adjoint part to logicism; in another chapter, one of them should have to treat the foundations of formalism (in particular the consistency of the calculus). In June 1932 (letter 4), Heyting proposed to Gödel to write a part on metamathematics: they would have later decided whether to insert it in the chapter about logicism or, better, in that about formalism. In any case, he left Gödel a wide treatment of Poincaré's criticisms in general, and suggested also to devote him a separate chapter, because Poincaré's criticisms concerned many foundational viewpoints.

Heyting stated that there were works that treated formal logic “without philosophical presuppositions”, for instance American investigations about axiomatics of Boolean algebra, the works of Bernays and Schönfinkel, the works about the Entscheidungsproblem, and Hilbert’s “Theoretische Logik”. He suggested to put them “im Anschluss an dem Logizismus”; he also left to Gödel the presentation of Chwistek (either among the logisticians or in the chapter that collected all other viewpoints) and the relationship between logicism and the natural sciences (to be inserted in the final chapter).

Gödel in fact did not send his part. Heyting took care of the part devoted to both "intuitionism " and "formalism" (called inside the book: “Axiomatik und Beweistheorie”) and wrote other two parts around “Andere Standpunkte” and “Mathematik und Naturwissenschaft”. In “Andere Standpunkte” he affirmed that never two mathematicians agree completely about that subject. Furthermore, also many philosophers turned their attention to mathematics. Therefore, a lot of foundational nuances could be listed. Still, even if they could be embraced all in the book, the author could not discuss them “because the arguments of philosophers are mostly understandable only in the context of their systems.” (p.67)

Consequently, he only quoted Otto Hölder (“his book contains an abundance of important individual remarks, but it does not arrive at an own point of view”), Trosten Brodén (“his work could not stand a serious criticism, at least for what concerns its mathematical aspects”), Julius König (“his book contains both philosophical and mathematical remarkable researches. Even before Hilbert, he developed a purely formal system [...] however, the formal system is very little comprehensible, and these considerations take up only a very small part of the book” p. 57) and, for France, Meyerson (“In France, a school has been formed which considers mathematics from the empirical point of view - in the usual philosophical sense, not in the sense of Borel -. An overview of this can be found in Meyerson, who himself wants to occupy a mediating position between empiricism and a priorism p. 58). The page devoted to Mannoury had been written by Mannoury himself, and at the end Pasch’s so called “empirism” was quickly described.

In his 1934 book he showed that he had acknowledged the tri-partition of the foundational schools, but he showed at the same time some oscillation in terminology, an openness about the possibility of other schools and, especially in the letters where he and Gödel had to establish who would write what, some doubts in attaching labels to research projects.

In his 1949 “Spanningen in de wiskunde”, his inaugural address, Heyting described a formal attitude in the foundations of mathematics tempted by Frege and then by Hilbert. He did not use either “logistic” or “logicism” for distinguishing among them. He affirmed that the axiomatic method (that accompanied our history from Euclid on) could not give a foundation for mathematics, because axioms themselves required a justification. Therefore axiomatics put the foundational questions and answered in formalistic terms: “(Hilbert is then arrived quite soon to a formalistic standpoint)” (p. 9 – 458).

He added that Frege had already arrived to a formalistic construction of mathematics, in which mathematics was considered a part of logic. His system had produced antinomies, therefore Hilbert constructed his own, by keeping logic and mathematics in parallel in the same system, looking for a proof of non-contradictoriness. Gödel proved that such proof cannot be found inside the system itself or in a weaker one. In the meantime Brouwer had given an intuitionistic foundation of mathematics.

At the end he put the question whether (and why) intuitionistic mathematics was worthwhile. He doubted that it could be useful in natural sciences, still, if intuitionism would be put in fruitful relationship with other “denkvormen”, then high results were to be expected for what concerned both logic and the theory of knowledge. Heyting put the question of worthiness, by keeping as an alternative the “klassieke school”. This expression explains why he used only the two labels “formalism” and “intuitionism”: he saw intuitionistic mathematics (and logic) as very different from the unique preceding mathematics (and logic). *Therefore, he considered only two contraposing groups: the ancient (where the foundational question has begun and has been answered in formalistic terms) and the new (intuitionism).*

In “Sur la tâche de la philosophie des mathematiques” (1953) he spoke of “supporters of logistics”, “mathematicians believing that rigor is found only in the manipulation of formulas ” and “intuitionists” as “the more modern currents of thought”, but at the end of the paper he made a *comparison only between the viewpoints of intuitionists and that of formalists as representing (at a deeper insight) the two aspects which classical mathematics consisted of*. He suggested, as a task for the philosopher of science, to point out that both the formalists and the intuitionists just developed one aspect of classical mathematics; they could not stand by themselves, each separated from the other, but they took care of aspects, none of which could be detached from mathematics:

“a formalist mathematician writes formulas according to clearly formulated rules, but the philosopher will focus also on the value judgements that the formalist does in his system, and on the tacit interpretation that guides him in the construction of the system. Similarly, when the intuitionist pretends to be interested exclusively in mental constructions, the philosopher will remark that he himself had made formal calculations and wonder what is the role of the calculation inside the intuitionistic viewpoint” (1953, p. 195).

We see that Heyting did not focus his attention on logicism: he always left it out of his deeper analysis.

In his 1956 “Intuitionism: an Introduction”, Heyting proposed at the very beginning a dialogue among a classical logician, a formalist, an intuitionist, a “letteralist”, a pragmatist, and a representant of the Signific. The dialogue was centered around the possible criticisms to intuitionism, but the criticisms were expressed by well specified “sources” that – in some cases – described also their own position. No “logicist” appeared under this label.

The “classicist” pointed out the following criticisms:

1) Intuitionistic mathematics could not be seen as the whole mathematics but only as a part of classical mathematics ;

2) Intuitionistic mathematics believed to do without logic, but it built castles in the air if it would not have the firm ground that only logic could offer ;

3) Intuitionistic mathematics presented “truths” that were not absolute (i.e. eternal, valid in any time), but that began to be valid at the moment in which, for instance, a certain object was build satisfying a given property.

The “letteralist” defined his viewpoint about mathematics: “Mathematics is quite a simple thing. I define some signs and I give some rules for combining them; that is all.”. He did not need proof of consistency for their formal systems because these were directly confronted with applications and in general they proved to be useful. This would be “difficult to explain if every formula were deducible in them” (p. 7). He criticized intuitionism for the following reasons:

1) Intuitionism accepted the infinite in mathematics (even if in its potential form), but clarity could be reached only by remaining in the finite .

2) Intuitionism had a dogmatic character because it accepted some principles (for instance, complete induction) and refused some others (for instance, the principle of excluded middle), even if most people considered all of them as evident .

3) Intuitionism had a “theological character” because mathematical intuition inspired them with objective and eternal truths.

The “significist” criticized intuitionism for its reluctance towards formalization, that was the ideal of modern scientist and that was, according to him, the only access to mental constructions .

The “pragmatist” agreed with significists and added: “The ideal of modern scientist is to prepare an arsenal of formal systems ready for use from which he can choose, for any theory, that system which correctly represents the experimental results”.

The formalist criticized intuitionism for the following reasons:

1) Neither terms nor derivation rules were well defined: therefore the risk of misunderstanding was high .

2) It destroyed a large part of classical mathematics .

It is very interesting that the intuitionist stated at the very beginning (p. 1) that mental mathematical constructions were objects that required an “own” logic, i.e. intuitionistic logic. We can interpret this as an application of Carnap’s principle of tolerance. Heyting referred it later, by letting it be quoted by the so called formalist. Namely, when the intuitionist affirmed that the difference between formalists and intuitionists was mainly one of tastes, the formalist replied : “If you will not quarrel with formalism, neither will I with intuitionism. Formalists are among the most pacific of mankind. Any theory may be formalized and then becomes subject to our methods. Also intuitionistic mathematics may and will be treated” (1956, p. 4). The direct reference to Carnap 1934 and 1937 was put inside this quote.

In 1958 “Blick von der intuitionistischen Warte” he stated that classical mathematics was a “strange mixture of very heterogeneous components” (p. 338). Intuitionism and formalism had pointed out two of those aspects, i.e. the formal side and the part that was grounded on number-intuition. They were in error when each believed to be “the only right one”: “formal mathematics always contains a rest of intuition, while intuitionist mathematics does not come without the use of formulas”. Still, there was a third direction, the platonistic one, consisting in the belief in the existence of a world of mathematical objects. It was refused by the first two directions, “nevertheless, the majority of mathematicians insist on this view; they use the classic methods of proof without considering them as purely formal developments”.

In 1958 “On truth in mathematics” he presented again a tripartition of classical mathematics, by specifying that the component that he “should like to call the naïve, but which was often called the platonic one” (p. 277 – 574) had been shared by the mathematicians till 1900, then it was put in doubt by the discovery of paradoxes, by Hilbert and by Brouwer. According to Heyting, the assumption of an abstract reality of any sort was meaningless, still “It seems to be increasing nowadays, under the influence of the successes of the semantic school, under the leadership of Tarski”. (p. 278 – 575)

In 1962, as we expect from the title “After thirty years”, Heyting referred to the Königsberg Conference, hence he mentioned logicism, even if he then let aside both logicism and formalism to devote himself to discuss intuitionism: “None of the conceptions of mathematics is today as clear-cut as it was in 1930. I shall be short about formalism and logicism. Formalism is the least vulnerable, but for metamathematical work it needed some form of intuitive mathematics. As to logicism, many axiomatic systems of logic and of set theory compete. It has proved not to be intuitively clear what is intuitively clear in mathematics. It is even possible to construct a descending scale of grades of evidence” (1963, p. 195). He stated that it had been recognized that there were "*intuitive, formal, logical and platonic*” elements inside mathematics.

In his 1968 paper "Wijsbegeerte de wiskunde" (“Philosophy of mathematics”, accessible, of course, only to Dutch readers), Heyting offered a review that started by Cantor and Frege, included Hilbert and Brouwer but also Mannoury and a brief reference to E.W. Beth. As for Beth, he stressed that his philosophy of mathematics was more a program than a complete theorization, consisting of a conceptual realism that distinguished four spheres of reality (the physical reality – the world of matter; the social reality – the world of men; the subjective reality – the world of the spirit; the logical reality – the world of “redenering”). Mathematics would belong to the last sphere.

In 1974, in presenting “Intuitionistic views about the nature of mathematics”, Heyting wrote only about intuitionism and formalism by stating that intuitionism described mathematical thought while formalism could only offer a linguistic structure. In this context he compared intuitionism to “the most radical form of formalism”, that he described in the following way: “The formalist considers every intuitive mathematical reasoning as inexact. He studies the language in which such reasonings are expressed and tries to formalize them. The result is a formal system consisting of a finite number of symbols and a finite number of rules for combining them into formulas” (p. 89 – 753).

The result of the comparison was: “There is no conflict between intuitionism and formalism when each keeps to its own subject, intuitionism to mental constructions, formalism to the construction of a formal system, motivated by its internal beauty or by its utility for science and industry. They clash when formalists contend that their systems express mathematical thought. Intuitionists make two objections against this contention. In the first place, as I have argued, just now, mental constructions cannot be rendered exactly by means of language; secondly the usual interpretation of the formal system is untenable as a mental construction” (ibidem). In this perspective, logic would be either a part of mathematics (if we interpreted syllogisms in term of set theory) or applied mathematics (if we interpreted syllogisms in terms of truth-value of linguistic expressions).

The paper ended with a gradualist vision of the various aspects of culture. At the lowest (and commonest) level there was the creation of a finite number of individual entities and the relations between them. The mathematical systems used in modern physics were enormously more refined than those that were at the basis of history, but also the work of the historian consisted in establishing relations between the facts that he had isolated in the continuous stream of events.

In "History of the foundations of mathematics" of 1978, which covered a time horizon till 1940, Heyting described the basic steps of the subject (formalization of logic, the paradoxes of set theory, type theory, proof theory) and the so-called ‘dramatic events’ (the discovery of Russell's paradox and Gödel's theorems); at the end, he presented the three Dutch figures of Brouwer, with two final chapters devoted to intuitionistic topics (choice sequences and “the continuum as a spread”), Beth, Mannoury (both of them very briefly).

He concluded with his observations about the situation of intuitionism at that time: “The editors of this collection decided to delimit it to the years before 1940. Still I cannot end without remarking that afterwards the situation has completely changed. It is generally recognized that intuitionism makes sense and that it is worth while to study it. The controversy between intuitionism and formalism has been solved. Dutch workers on foundations are no longer isolated; they collaborate intensively with their colleagues all over the world.”

It is clear from all these quotes that Heyting firmly believed that intuitionism was the right foundation of mathematics. Still, he was very respectful of others' opinions and believed that collaboration among mathematicians was highly relevant. He was open to new ideas, new results, new projects. Furthermore, it is also clear from the analysis of all his foundational papers that Heyting did not see the foundational schools as three clear-cut stones, but he observed in the foundational panorama many suggestions that could or could not fit with an existent school. He felt that the suggestions could be presented under the name of the proponent or under an old label or a newly coined one.

I think that the two aspects (openness to others' point of view and freedom in labelling) were linked: his frequent changes in presenting the foundational schools, his description of a broad range of them and his freedom in labelling were signs of his openness to the others' points of view and of his will of avoiding rigid contrapositions among them. By contrast, Brouwer constant use of only two labels for describing the foundational panorama was a sign of his will of absolutizing his viewpoint: all the possible nuances of other perspectives were collected under a unique label in the perspective of a final duel that should have led to a definitive victory.

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