A generalized Chang completeness theorem

A. Di Nola, G. Lenzi and L. Spada (Salerno)
V. Marra (Milano)

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The original Chang completeness theorem

Theorem

The variety of MV-algebras is generated by $[0, 1] \cap Q$.

The main theorem

Theorem 1

Let C be a product MV-algebra in the sense of Montagna (JLLI 2000). Assume also that C is totally ordered and torsion free. Let R be the ring such that $\Gamma(R, 1) = C$, let R^q be the quotient field of R and $C^q = \Gamma(R^q, 1)$. Then the variety of MV_C-algebras is generated by C^q.

Note that our theorem implies Chang theorem by letting $C = \{0, 1\}$.

MV-algebras

Recall that an MV-algebra is a structure $(A, \oplus, \ominus, 0, 1)$ such that:

- $(A, \oplus, 0)$ is a (commutative) monoid;
- $x \oplus 1 = 1$;
- $1 \ominus (1 \ominus x) = x$;
- $x \oplus (1 \ominus x) = 1$;
- $x \ominus 0 = x$;
- $x \oplus (y \ominus x) = y \oplus (x \ominus y)$;
- $x \ominus x = 1 \ominus ((1 \ominus x) \oplus y)$.

This presentation is due to Montagna and is slightly nonstandard.

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Example

We can consider

- $A = [0, 1]$;
- $x \oplus y = min(1, x + y)$;
- $x \ominus y = max(0, x - y)$.

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Product MV-algebras

A product MV-algebra is a structure $(A, \oplus, \ominus, \cdot, 0, 1)$ satisying the axioms of MV-algebra and the following:

- $\cdot$ is an associative commutative binary operation;
- $x \cdot 1 = x$;
- $x \cdot (y \ominus z) = xy \ominus xz$.

(Sometimes the dot $\cdot$ is omitted).

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MV_C-algebras

Let C be a product MV-algebra. An MV_C-algebra is an MV-algebra equipped with unary operators $f_c, c \in C$ such that

- $f_a(x \ominus y) = f_a(x) \ominus f_a(y)$;
- $f_{a \ominus b}(x) = f_a(x) \ominus f_b(x)$;
- $f_a(f_b(x)) = f_{ab}(x)$;
- $f_1(x) = x$.

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Lattice ordered groups and rings

Let G be a lattice ordered group (or ring). A strong unit is an element $u \in G$ such that for every $x \in G$ there is $n \in N$ such that $x \leq nu$.

The Mundici Γ functor

There is a bijection Γ between lattice ordered groups with strong unit and MV-algebras. We have $\Gamma(G, u) = [0, u]$ where $x \oplus y = u \wedge (x + y)$ and $x \ominus y = 0 \vee (x - y)$.
Likewise there is a bijection (still called Γ) between lattice ordered commutative rings with strong unit and product MV-algebras. In fact, $\Gamma(R, u) = [0, u]$ and the product on $[0, u]$ is the restriction of the one of R.
Finally, if C is a product MV-algebra with $\Gamma(R, u) = C$, there is a bijection (still called Γ) between lattice ordered R-modules with strong unit and MV_C-algebras.

Torsion

A product MV-algebra is said to have torsion if there are $a, b \neq 0$ such that $ab = 0$. Similarly one speaks of torsion of an MV_C-algebra, ring or module.

Beginning of the proof of Theorem 1

When $C = \{0, 1\}$ it is the classical Chang completeness theorem, so we will suppose $C \neq \{0, 1\}$.
Suppose an MV_C-algebra A verifies $\exists a.t(a) \neq 0$, where t is a term in the language of MV_C-algebras.
We must show that C^q also verifies $\exists a.t(a) \neq 0$.
By Chang subdirect representation theorem we can suppose that A is totally ordered.

If $A = \{0, 1\}$ then also $A' = \{0, 1/2, 1\}$ has a structure of MV_C-algebra, because we can define $f_c(1/2) = f_c(1)/2$, and A' has an element different from $0, 1$, and $t(a) = 1$ in A'.
So we can suppose that in A there is an element $b \neq 0, 1$.
If $t(a) = 1$ then we can consider $t'(a, b) = t(a) \ominus b$, noting that $0 < t'(a, b) < 1$.
In other words we can suppose A totally ordered and $0 < t(a) < 1$ for some $a \in A$.

Note that C is commutative (by definition of product MV-algebra), totally ordered and torsion free; hence, the ring R is a totally ordered, commutative integral domain (see Montagna op. cit.).
Let $M(A)$ be the R-module such that $\Gamma(M(A), u) = A$.

A problem is that A and $M(A)$ could have torsion, but we can solve this problem as follows.

From $M(A)$ to M'

Let $t'(u, a)$ be a term over lattice ordered R-modules such that

$$(M(A), u) \models t'(u, a) = t(a).$$

Hence $(M(A), u) \models 0 < t'(u, a) < u$.
Now let ϵ be a constant (considered infinitesimal) and let M' be the R-module of the formal sums $m + \Sigma_{i=1}^{n} r_i \epsilon^i$, where $m \in M(A)$ and $r_i \in R$.
We order M' so that a formal sum is positive if and only if some coefficient is nonzero and the lowest nonzero coefficient is positive.
This is a total order on M' which extends the one on $M(A)$.
In particular, ϵ is smaller than any positive element of $M(A)$. We say that two elements x, y of M' are infinitesimally close if $|x - y| \leq r\epsilon$ for some $r \in R$.

From M' to M''

Suppose that t' contains the variables $u, a_1, \ldots, a_n$. Then

$$M' \models 0 < t'(u+\epsilon, a_1+\epsilon^2, \ldots, a_n+\epsilon^{n+1}) < u.$$

In fact, the element

$$t'(u+\epsilon, a_1+\epsilon^2, \ldots, a_n+\epsilon^{n+1})$$

is infinitesimally close to

$$t'(u, a_1, \ldots, a_n)$$

which is between zero and one in $M(A)$, so the former is also between zero and one in M'.
Now consider the R-submodule M'' of M' generated by the set

$$S = \{u+\epsilon, a_1+\epsilon^2, \ldots, a_n+\epsilon^{n+1}\}.$$

M'' is torsion free

In fact, M'' is the set of formal sums

$$r_0(u+\epsilon)+\Sigma_i r_i(a_i+\epsilon^{i+1})$$

where $r_i \in R$.
Suppose $r \neq 0$ and $rx = 0$, where $x \in M''$:

$$x = r_0(u+\epsilon)+\Sigma_i r_i(a_i+\epsilon^{i+1}).$$

Then the infinitesimal part of the sum is also zero, so

$$r(r_0\epsilon+\Sigma_i r_i\epsilon^{i+1}) = 0.$$

By definition of M', since the latter sum is zero, all its coefficients are zero, so $rr_i = 0$ for every i; but R is an integral domain and $r \neq 0$, so $r_i = 0$ for every i, hence $x = 0$. Summing up, M'' is torsion free.
Moreover, we have

$$M'' \models \exists u > 0.\exists a.0 < a < u \wedge 0 < t'(u,a) < u.$$

Quantifier elimination

Since M'' is totally ordered and torsion free R-module, it can be embedded in a totally ordered, torsion free, R-divisible R-module M''' and we have

$$M''' \models \exists u > 0. \exists a. 0 < a < u \wedge 0 < t'(u, a) < u.$$

At this point a quantifier elimination argument can be applied, M''' is elementarily equivalent to R^q as a lattice ordered R-module, and the above formula holds in R^q:

$$R^q \models \exists u > 0. \exists a. 0 < a < u \wedge 0 < t'(u, a) < u,$$

that is, for some $u > 0$ in R^q we have

$$(R^q, u) \models \exists a. 0 < a < u \wedge 0 < t'(u, a) < u.$$

End of the proof

Now we have an automorphism ϕ of R^q as a lattice ordered R-module sending 1 to u: it is enough to take $\phi(x) = ux$. So, the formula above is true in $R^q, 1$ as well:

$$(R^q, 1) \models \exists a.0 < a < u \wedge 0 < t'(u, a) < u$$

and passing to the Γ functor, we conclude

$$C^q = \Gamma(R^q, 1) \models \exists a.t(a) \neq 0.$$

Summing up, C^q generates the variety of MV_C-algebras.

If you want to know more...

Antonio Di Nola: adinola@unisa.it

Giacomo Lenzi: gilenzi@unisa.it

Vincenzo Marra: Vincenzo.Marra@unimi.it

Luca Spada: Luca.Spada@gmail.com

Thank you!