The complexity of the classification problem in ergodic theory

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Its mathematical formalization can be traced back to the 1930s (von Neumann, Rokhlin)

Standard probability spaces

Definition

An atomless standard probability space $(X, \mathcal{B}, \mu)$ is a set X endowed with a σ-algebra $\mathcal{B}$ and a probability measure μ, which is isomorphic to the unit interval $[0, 1]$ endowed with the Borel σ-algebra and the Lebesgue measure.

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Let X be any locally compact space (or, more generally, a Polish space) endowed with the Borel σ-algebra $\mathcal{B}$ and an atomless measure μ. Then $(X, \mathcal{B}, \mu)$ is a standard probability space.

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Different presentations of the standard probability space are useful to produce examples

Measure-preserving automorphisms

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This is in fact a Polish group with respect to the topology given by setting

$$T_i \to T \quad \text{if and only if} \quad \|f \circ T_i - f \circ T\|_2 \to 0$$

for every $f \in L^\infty(X, \mathcal{B}, \mu)$.

Irrational rotation

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If $\theta \in [0,1]$ is an irrational number, then the map

$$t \mapsto \exp(2\pi i\theta)\, t$$

is an automorphism of $\mathbb{T}$.

Measure-preserving actions

Let Γ be countable discrete group.

Definition

A Γ-action on $(X, \mathcal{B}, \mu)$ or Γ-dynamical system is a group homomorphism

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Example (Bernoulli shift)

Consider $[0,1]^{\Gamma}$ with the product measure. The Bernoulli action $g \mapsto \beta_g$ of Γ on $[0,1]^{\Gamma}$ is defined by setting, for $g \in \Gamma$ and $(t_h)_{h \in \Gamma} \in [0,1]^{\Gamma}$,

$$\beta_g \left(t_h\right)_{h\in\Gamma} = \left(t_{gh}\right)_{h\in\Gamma} .$$

Conjugacy of actions

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This can be seen as a minimality condition, saying that the action can not be decomposed into simpler actions.

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Definition

A Γ-action α is free if, for every nonidentity element g of Γ, the set of fixed points of α_g is null.

Freeness is a nondegeneracy condition, which in particular ensures that the action is faithful.

Free ergodic actions are, in some sense, the basic building blocks of more complicated actions.

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Thus any group admits a free ergodic action.

Classification in ergodic theory

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Example

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Thus any group admits a free ergodic action.

Classification of free ergodic action is a central problem since the early days of ergodic theory

Problem (Halmos, 1956)

For a fixed Γ, is there an explicit way to classify free ergodic Γ-actions?

Classification in ergodic theory

One should clarify the notion of classification to make the question precise.

Definition

Two Γ-actions α, α' on $(X, \mathcal{B}, \mu)$ are conjugate if there is $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ such that $T \circ \alpha_g = \alpha'_g \circ T$ for every $g \in \Gamma$.

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An explicit classification of free ergodic Γ-actions up to conjugacy is an effective procedure that allows one to tell whether two such actions are conjugate or not.

Nonclassifiability for conjugacy

The space $\mathrm{FrErg}_\Gamma(X, \mathcal{B}, \mu)$ of free ergodic Γ-actions is endowed with a canonical Polish topology, given by identifying it as a subspace of $\mathrm{Aut}(X, \mathcal{B}, \mu)^\Gamma$ endowed with the product topology.

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The following is a possible precise reformulation of Halmos' problem:

Problem

Is the relation of conjugacy of free ergodic Γ-actions

$$\{(\alpha, \alpha') : \alpha \text{ and } \alpha' \text{ are conjugate free ergodic actions}\}$$

a Borel set in the product space $\mathrm{FrErg}_\Gamma(X, \mathcal{B}, \mu) \times \mathrm{FrErg}_\Gamma(X, \mathcal{B}, \mu)$ endowed with the product topology?

Nonclassifiability of conjugacy

Theorem (Foreman–Rudolph–Weiss, 2011)

The relation of conjugacy of free ergodic $\mathbb{Z}$-actions is not a Borel set.

It is conjectured that the same holds for any infinite group.

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Theorem (Gardella–L., 2017)

If Γ is a nonamenable group, then the relation of conjugacy of free ergodic Γ-actions is not a Borel set.

The proof in the nonamenable case is very different from the case of $\mathbb{Z}$.

Orbit equivalence

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In the case of amenable groups, it is much coarser.

Theorem (Dye 1959, Ornstein–Weiss 1987)

Let Γ be an amenable countable group. All the free ergodic Γ-actions are orbit equivalent.

The number of orbit equivalence classes

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This was shown by

- Gaboriau–Popa when $\Gamma = \mathbb{F}_2$ (2005),
- Ioana when Γ contains a copy of $\mathbb{F}_2$ (2011), and
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These results motivated the following question:

Problem (Kechris, 2010)

Let Γ be a nonamenable group. Is the relation of orbit equivalence of free ergodic Γ-actions a Borel set?

Nonclassifiability of orbit equivalence

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In the case when Γ contains $\mathbb{F}_2$ as a normal subgroup, this was shown by Epstein–Törnquist (2012).

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A sufficient criterion

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(1): guaranteed as long as the construction of α_A from A is functorial.
(2): need to attach to Γ-actions an invariant that can capture the group A.

Cohomology groups and conjugacy invariants

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The scope of these results was significantly extended in the past 15 years with the infusion of methods from operator algebras (Popa, Ioana, Peterson, Chifan)

Actions with prescribed cohomology

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For this an additional sort of rigidity is required.

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Given an $\mathbb{F}_2$-action α on X one can consider the product action $\alpha \times \rho$ on $X \times \mathbb{T}^2$ defined by

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The action ρ satisfies the following rigidity property (Popa):

- the orbit equivalence class of the action $\alpha \times \rho$ "remembers" the conjugacy class of α up to countable sets.

The proof in the case of $\mathbb{F}_2$

One obtains an assignment $A \mapsto \alpha_A \times \rho$ from countable abelian groups to free ergodic $\mathbb{F}_2$-actions that satisfies the lemma for $\Gamma = \mathbb{F}_2$.

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The work lies in showing that this satisfies the hypothesis of the lemma

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Theorem (Gaboriau–Lyons, 2009)

Let β be the Bernoulli Γ-action on $[0,1]^\Gamma$. There exists a free ergodic $\mathbb{F}_2$-action θ on $[0,1]^\Gamma$ such that, up to discarding a null set, θ-orbits are contained in β-orbits or, equivalently, $R(\theta) \subset R(\beta)$.

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Epstein: given an $\mathbb{F}_2$-action α one can define the co-induced action

$$\mathrm{CInd}_{R(\theta)}^{R(\beta)}(\alpha)$$

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Consider the assignment from abelian groups to Γ-actions

$$A \mapsto \mathrm{CInd}_{R(\theta)}^{R(\beta)} (\alpha_A \times \rho)$$

The core of the proof is to show that it satisfies the hypotheses of the main lemma.

The groupoid perspective

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This concludes the proof of:

Theorem (Gardella–L., 2017)

If Γ is a nonamenable group, then the relation of conjugacy of free ergodic Γ-actions is not a Borel set.

In fact, we obtain a more general version of the theorem for actions of nonamenable groupoids

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Theorem (Dye 1959, Connes–Feldman–Weiss 1981)

If G is amenable, then all the free ergodic G-actions are orbit equivalent.

Nonamenable locally compact unimodular groups

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In fact, one needs the more general version for actions of groupoids

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A noncommutative space is an algebra $\mathcal{A}$ of operators on a Hilbert space which is invariant under taking adjoints and it is closed in the topology given by the operator norm (noncommutative topological space) or even the topology of pointwise convergence (noncommutative measure space)

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The noncommutative setting is in some sense richer, as there exist many natural analogues of the standard probability space

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This family includes various algebras. The easiest to describe are the UHF C*-algebras, which are direct limits of matrix algebras completed with respect to the operator norm.

A conjecture

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- in the nonamenable case, when $\mathcal{A}$ is UHF and Γ is "rigid" (Gardella–L., 2016).

Future work

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This will involve initiating the study of cocycle superrigidity for strongly self-absorbing C*-algebras, which is of independent interest.