The isomorphism relation of classifiable shallow theories

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Our object of study

The theories that we consider:

- are formulated in a *countable first-order language*;
- are *complete* (models are logically equivalent);
- have an *infinite model* (and thus, by Löwenheim-Skolem, a model of each infinite size).

Complexity of a theory

Several approaches:

1. decomposition of models (Classification Theory)
2. number of nonisomorphic models (spectrum analysis)
3. set-theoretic complexity of the isomorphism relation (Descriptive Set Theory)

First approach: Classification Theory

Dichotomies:

- stable vs unstable
- superstable vs unsuperstable
- NDOP vs DOP (dimensional order property)
- NOTOP vs OTOP (omitting types order property)
- shallow vs deep

Classifiable theories are (stable) superstable NDOP NOTOP.

Uncountable models of classifiable shallow theories admit a “nice” decomposition into well-founded trees of countable submodels. The maximum depth of such decompositions is the *depth* of the theory.

Examples

- ACF_0 and $\mathrm{Th}(\mathbb{Z}, +, -, 0)$ are classifiable shallow of depth 1.
- The theory of α infinitely recoarsing equivalence relations is classifiable shallow of depth $\alpha + 1$.
- The theory of a single unary function such that each element has infinitely many preimages is superstable deep.
- $\mathrm{Th}(\mathbb{Z}^\omega, +, 0)$ is stable unsuperstable.
- $\mathrm{Th}(\mathbb{Z}, +, \cdot, 0, 1)$ and DLO are unstable.

Second approach: spectrum analysis

The *spectrum function* $I(\kappa, T)$ gives the number of nonisomorphic models of T of size κ. In general $1 \leq I(\kappa, T) \leq 2^\kappa$.

Examples:

- $I(\kappa, \mathrm{ACF}_0) = \begin{cases} \omega & \kappa = \omega \\ 1 & \kappa > \omega \end{cases}$
- $I(\kappa, \mathrm{DLO}) = \begin{cases} 1 & \kappa = \omega \\ 2^\kappa & \kappa > \omega \end{cases}$

Looking at the uncountable spectrum, one could argue that ACF_0 is simpler than DLO.

Shelah's Main Gap

The following result is the link between number of uncountable models and Classification Theory.

Theorem (Main Gap)

Let $\kappa > \omega$ be the γ-th cardinal. Then:

- *if T is classifiable shallow of depth α,*

$$I(\kappa, T) \leq \beth_\alpha \left(|\gamma|^{2^\omega}\right);$$

- *if T is not classifiable shallow, then*

$$I(\kappa, T) = 2^\kappa.$$

This means: either a theory has the maximum number of models for every uncountable size, or it has very few and the upper bound depends on the depth.

Third approach: Descriptive Set Theory

The Cantor space 2^ω is the completely metrizable separable space of binary sequences of countable length.

The *generalized Cantor space* 2^κ is the space of binary sequences of length κ. The *bounded topology* on 2^κ is generated by the family of sets $\{N_p \mid p \in 2^{<\kappa}\}$ where

$$N_p = \{\eta \in 2^\kappa \mid \eta \restriction \mathrm{dom}(p) = p\}.$$

If $\kappa^{<\kappa} = \kappa$, this base has size κ.

Generalized Borel sets

The Borel hierarchy on 2^κ:

- $\mathbf{\Sigma}^0_0 = \mathbf{\Pi}^0_0 = \{\text{clopen sets}\}$
- for every $1 \leq \alpha < \kappa^+$

$$\mathbf{\Sigma}^0_\alpha = \left\{ \bigcup_{i<\kappa} A_i \mid A_i \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta \right\}, \quad \mathbf{\Pi}^0_\alpha = \left\{ \bigcap_{i<\kappa} A_i \mid A_i \in \bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta \right\}$$

The *Borel rank* of a set A is the lowest α such that $A \in \mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha$.

Differences with standard DST

- $A \subseteq 2^\kappa$ is *analytic* ($\mathbf{\Sigma}^1_1$) if it is a continuous image of a closed subset of κ^κ.
- $A \subseteq 2^\kappa$ is *bianalytic* ($\mathbf{\Delta}^1_1$) if both A and $2^\kappa \setminus A$ are analytic.

	$\kappa = \omega$	$\kappa > \omega$
topology	bounded = product	bounded $\neq$ product
hierarchy	$\text{Borel} = \mathbf{\Delta}^1_1 \subset \mathbf{\Sigma}^1_1$	$\text{Borel} \subset \mathbf{\Delta}^1_1 \subset \mathbf{\Sigma}^1_1$

Coding models

The space of structures of size κ is

$$\prod_{i<\omega} 2^{\kappa^{n_i}} \approx 2^{\kappa}.$$

Thus the space of models of T is embeddable in 2^{κ}, and the isomorphism relation $\cong^{\kappa}_{T}$ between models can be seen as an analytic subset of $(2^{\kappa})^2 \approx 2^{\kappa}$.

This allows us to study the complexity of $\cong^{\kappa}_{T}$ (for example, whether it is Borel or not), which is yet another way to look at the complexity of T.

The countable case

When $\kappa = \omega$, the complexity of the isomorphism relation does not agree with classification theory.

Example: DLO is unstable, while $\cong^{\omega}_{\mathrm{DLO}}$ is trivial.

This is unsurprising as Shelah's Main Gap fails for $\kappa = \omega$ as well.

Link with Classification Theory

Theorem (S. D. Friedman, T. Hyttinen, V. Kulikov)

Let $\kappa^{<\kappa} = \kappa > 2^\omega$. Then $\cong^\kappa_T$ is Borel if and only if T is classifiable shallow.

(*Generalized descriptive set theory and classification theory*, Memoirs of the AMS (2014), vol. 230, no. 1081)

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Question: is there a relation between the Borel rank of $\cong^\kappa_T$ and the depth of T?

Our Main Gap

Theorem (Descriptive Main Gap)

Let $\kappa^{<\kappa} = \kappa > 2^\omega$.

- *If T is classifiable shallow of depth α, then $\cong^\kappa_T \in \mathbf{\Pi}^0_{4\alpha+2}$;*
- *if T is not classifiable shallow then $\cong^\kappa_T$ is not Borel.*

In particular, T is classifiable shallow if and only if $\cong^\kappa_T$ has countable rank.

Thus the Borel rank is either very small (depth providing an upper bound) or infinitely high (i.e. the relation is not Borel).

This Gap, unlike Shelah's, is never trivial for a fixed κ and in fact does not depend at all on the size κ of the models considered.

Scott height

$\mathcal{L}_{\kappa^+\kappa}$ is the extension of first order logic allowing κ-conjunctions, κ-disjunctions and $<\kappa$-quantifications. The complexity of an $\mathcal{L}_{\kappa^+\kappa}$-formula is classified by *quantifier rank*.

$\mathcal{M} \equiv_\beta \mathcal{N}$ means they verify the same $\mathcal{L}_{\kappa^+\kappa}$-formulas of rank $< \beta$. The smallest β such that $\mathcal{M} \equiv_\beta \mathcal{N} \Rightarrow \mathcal{M} \cong \mathcal{N}$ for every $\mathcal{M}, \mathcal{N}$ is the $\mathcal{L}_{\kappa^+\kappa}$-*Scott height* of *T*.

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Theorem (Shelah)

Let $\kappa > 2^\omega$ regular. Then β exists if and only if T is classifiable. Furthermore:

- *if T is classifiable shallow of depth α, then $\beta \leq 2\alpha$;*
- *if T is classifiable deep, then $\beta = \kappa^+$.*

Main result

Theorem

Let $\kappa^{<\kappa} = \kappa$. Suppose $\cong^{\kappa}_{T} \in \mathbf{\Pi}^0_\delta \setminus \bigcup_{\gamma<\delta} \mathbf{\Pi}^0_\gamma$. Let β be the $\mathcal{L}_{\kappa^+\kappa}$-Scott height of T. Then:

1. $\delta \leq 2\beta + 2$;
2. $\beta \leq \max\{3, \delta + 1\}$.

In particular, β and δ have finite distance.

Corollary (Descriptive Main Gap)

Let $\kappa^{<\kappa} = \kappa$.

- *If T has $\mathcal{L}_{\kappa^+\kappa}$-Scott height $\beta < \kappa^+$, then*

$$\cong^{\kappa}_{T} \in \mathbf{\Pi}^0_{2\beta+2}.$$

- *Otherwise, $\cong^{\kappa}_{T}$ is not Borel.*

Main result

Theorem

Let $\kappa^{<\kappa} = \kappa > 2^\omega$. Suppose $\cong^\kappa_T \in \mathbf{\Pi}^0_\delta \setminus \bigcup_{\gamma<\delta} \mathbf{\Pi}^0_\gamma$. Let β be the $\mathcal{L}_{\kappa^+\kappa}$-Scott height of T and let α be the depth of T. Then:

1. $\delta \leq 2\beta + 2 \leq 4\alpha + 2$;
2. $\beta \leq \max\{3, \delta + 1\}$.

In particular, β and δ have finite distance.

Corollary (Descriptive Main Gap)

Let $\kappa^{<\kappa} = \kappa > 2^\omega$.

- *If T has $\mathcal{L}_{\kappa^+\kappa}$-Scott height $\beta < \kappa^+$, then*

$$\cong^\kappa_T \in \mathbf{\Pi}^0_{2\beta+2} \subseteq \mathbf{\Pi}^0_{4\alpha+2} \subseteq \mathbf{\Pi}^0_{\omega_1}.$$

- *Otherwise, $\cong^\kappa_T$ is not Borel.*

Tools used in the proof

- Ehrenfeucht-Fraissé games
- Borel* sets
- Generalized Lopez-Escobar Theorem

Ehrenfeucht-Fraissé games

For every tree t and $\mathcal{M}, \mathcal{N}$ models we have an *Ehrenfeucht-Fraïssé game* $\mathrm{EF}_t^\kappa(\mathcal{M}, \mathcal{N})$ with the following rules. At every step:

- player I picks a node of t and a small subset $A \subset \kappa$;
- player II extends to A the domain of a small partial map $f : \kappa \to \kappa$.

The tree t works as a timer: the game ends when I arrives at the end of a branch. Then II wins if the resulting map is a partial isomorphism between $\mathcal{M}$ and $\mathcal{N}$, otherwise I wins.

EF games and Scott height

For every ordinal α we can define the tree t_α of all descending sequences in α, ordered by end extensions, with the empty sequence $\emptyset$ being the root.

Theorem

For any two models $\mathcal{M}, \mathcal{N}$ of size κ

$$\mathcal{M} \equiv_\alpha \mathcal{N} \Leftrightarrow \text{ II wins } \mathrm{EF}^\kappa_{t_\alpha}(\mathcal{M}, \mathcal{N}).$$

Thus the Scott height can be defined in terms of EF games.

Borel* sets

For every $\eta \in 2^\kappa$, t tree and

$$h : \{\text{branches of } t\} \to \{\text{clopen sets}\}$$

we have a game $G(t, h, \eta)$ where, starting from the root, players I and II alternate in picking a successor (on a limit round, I picks a successor of the supremum of all previous moves).
The game ends when either player arrives at the end of a branch b.
Player II wins if $\eta \in h(b)$; otherwise player I wins.

The set $\{\eta \mid \mathrm{II} \text{ wins } G(t, h, \eta)\}$ is a *Borel** set coded by (t, h).

Borel* and Borel

Theorem

Borel sets in $\mathbf{\Pi}^0_\delta$ are exactly the Borel sets coded by well-founded trees of rank $\delta + 1$.*

In standard DST ($\kappa = \omega$), Borel=Borel*.

Borel* sets and EF games

Theorem

If t has rank $\beta + 1$, there is a tree u_t of rank $2\beta + 3$ and a labeling h such that

$$\text{II wins } \mathrm{EF}_t^\kappa(\mathcal{M}, \mathcal{N}) \Leftrightarrow \text{II wins } G(u_t, h, (\mathcal{M}, \mathcal{N})).$$

If the left side equals $\mathcal{M} \cong \mathcal{N}$, then (u_t, h) is a code for $\cong_T^\kappa$. This leads to an upper bound on the Borel rank of $\cong_T^\kappa$ depending on the Scott height of T.

Generalized Lopez-Escobar Theorem

Theorem

Let $\kappa^{<\kappa} = \kappa$. A family of $\mathcal{L}$-structures is Borel and closed under isomorphisms if and only if is axiomatized by a sentence $\sigma \in \mathcal{L}_{\kappa^+\kappa}$ without parameters. Furthermore, if A has rank δ then we can find σ with quantifier rank δ.

The theorem can be used to define $\cong^{\kappa}_{T}$ (in a certain extended language), which shows that its Borel rank is an upper bound for the Scott height of T.

Beyond Borel

The following results hold in L, the universe of constructible sets.

Theorem (Friedman, Hyttinen, Kulikov)

Let $\kappa^{<\kappa} = \kappa > 2^\omega$ with uncountable cofinality. Then $\cong^\kappa_T$ is $\mathbf{\Delta}^1_1$ if and only if T is classifiable.

(*Generalized descriptive set theory and classification theory*, Memoirs of the AMS (2014), vol. 230, no. 1081)

Theorem (Hyttinen, Kulikov)

There exists a stable unsuperstable NDOP NOTOP theory with an isomorphism relation that is $\mathbf{\Sigma}^1_1$-complete, i.e. every isomorphism relation is reducible to it.

(*On Σ^1_1-complete equivalence relations on the generalized Baire space*, Math. Log. Quart. (2015), no. 61)

Thanks for the attention!