On Popper's Decomposition of Logical Notions

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METHODOLOGICAL ISSUES

- Popper's idea is to consider a number of languages and *translations* from one of these languages into the others. The presupposition is that we master, or have competence, of the languages involved.
- We remind that for Popper Logic is a *metalinguistic* enterprise. A distinctive feature of his approach, compared with now usual approaches, is that no *assumption* is made about the form or syntactic structure of the (object-)language, say $\mathcal{L}$. $\mathcal{L}$ could also be a formally defined language, but nothing excludes its being a natural language.

METHODOLOGICAL ISSUES II

- Popper's theory of inference intends to provide a tool which can be applied to any language in which we can identify statements, whatever their logical structure or lack of structure may be; that is to say, expressions of which we might reasonably say that they are true or that they are false.
- Popper starts by focusing on the *problem of giving a satisfactory definition of "valid deductive inference"*, where "deducibility" is the only undefined notion employed, as far as propositional and modal logic are concerned.

METHODOLOGICAL ISSUES III

- Popper presents a model consisting of an articulated structure given by inferential relations, which is to be laid on any $\mathcal{L}$, and aims at characterizing the meaning of logical compounds in the form of answers to questions like: "what does it mean for $\mathcal{L}$ to have an operation which has the inferential force of a negation, conjunction, . . . ?".
- This calls to mind the model of language which Quine was going to present shortly after (in 1951, with *Two Dogmas of Empiricism*): the model of an articulated structure, with some sentences lying at the periphery, where experience impinges, and others at varying levels within the interior. However, whereas Quine's proposal was driven by general meaning-theoretical issues, Popper just aimed at characterizing the meaning of logical notions.

METHODOLOGICAL ISSUES IV

We collect here some notions which will be soon useful. Let's consider two languages, say $\mathcal{L}_1$ and $\mathcal{L}_2$:

1. A translation of $\mathcal{L}_1$ into $\mathcal{L}_2$ such that every complete statement of the former is co-ordinated with one complete and meaningful statement of the latter is called an *interpretation*.
2. If the interpretaion preserves re-occurrences of statements then it is called a *statement-preserving interpretation*.
3. In the case in which with every different statement of $\mathcal{L}_1$ a different statement of $\mathcal{L}_2$ is coordinated, Popper speaks of a *strictly* statement-preserving interpretation.
4. In case a translation preserves the meaning of the statements of $\mathcal{L}_1$, it is called a *proper translation*.

METHODOLOGICAL ISSUES V

5. *Assuming* given the distinction between the *formative signs* and the *descriptive signs* of the languages we are considering, a *form-preserving interpretation* is now defined as an interpretation which
   - preserves the meaning of all the formative signs,
   - preserves recurrence of those groups of *descriptive* expressions which, in a proper translation, would fill the spaces between the translated formative signs.
6. Two statements a_1 and a_2, not necessarily belonging to the same language, have the same *logical form* if, and only if, there exist two form-preserving interpretations such that a_1 interprets a_2 and *vice versa*.
7. Then, the *logical form of the statement* a_1 is defined as the class of statements (of any number of languages) which have *the same logical form as* a_1.

METHODOLOGICAL ISSUES VI

8. The *logical skeleton* of a statement is obtained simply by eliminating all descriptive signs, and indicating, at the same time, recurrences of descriptive signs, by some method or other. Two statements a_1 and a_2, sharing the same "logical skeleton", belong to the <u>*same*</u> language.
9. <u>*Assuming*</u> given the distinction between the *formative signs* and the *descriptive signs* of the language, the notion of "logical skeleton" admits a direct definition, i.e. without passing through the idea of interpretation. On the other hand, and under the same <u>*assumption*</u>, the idea of a logical form is more general, and gives us the means of constructing a theory of language – or of languages – *without tying us down to any particular language*.

METHODOLOGICAL ISSUES VII

- Tackling the notion of a valid (deductive) inference, Popper tries various proposals:

  (D1) An inference is valid iff every possible state of affairs which renders all the premises true also renders the conclusion true.

  A reformulation of (D1) is obtained by exploiting the notion of *counter-example*:

  (D1') An inference is valid iff no counter-example of it exists

- Problems: "state of affairs"? "possible state of affairs"? does "possible" mean "*logically* possible"?
  A second proposal uses the notion of "logical skeleton":

  (D2) An inference is valid iff every inference with the same logical skeleton whose premises are all true has a true conclusion.

METHODOLOGICAL ISSUES VIII

- Another possibility is that we use the notion of "logical form" defined with the help of the term "form-preserving interpretation":

  (D3) An inference is valid iff every form-preserving interpretation of it whose premises are all true has a true conclusion.

- An intrinsic limit of (D2) is given by its referring to other arguments *of the same logical skeleton* as the argument in question and thereby confines its reference to other arguments *belonging to the same language*. (D2), moreover, is conditioned by the possible poverty in descriptive signs of the given $\mathcal{L}$: an invalid inference would appear as valid from the point of view of (D2), simply because no counterexample exists within $\mathcal{L}$.

METHODOLOGICAL ISSUES IX

- By its referring to all form-preserving interpretations, and therefore to *an unspecified number of different languages*, viz., to all those into which the formative signs can be properly translated, (D3) warrants that the validity or otherwise of an inference or rule of inference is independent of the language in which it is formulated.
- The problem with (D3) is that it is based on the distinction between formative and descriptive signs. A distinction that only “interpretation” and “statement-preserving interpretation” do not presuppose.

METHODOLOGICAL ISSUES X

- Popper proposes to start from a number of inferences which are valid *whatever the logical form of the statements involved.* Since they are valid indipendently of the distinction “formative/descriptive”, they are called *absolutely valid*.
- Having a certain system of absolutely valid rules at our disposal, it is possible to define the logical force or import of the various formative signs in terms of deducibility: this is called an *inferential definition*. Formative signs are characterized as those signs which can be given an inferential definition.
- The following definition is proposed:

  (D4) An inference is absolutely valid if, and only if, every statement-preserving interpretation whose premises are true has a true conclusion.

METHODOLOGICAL ISSUES X

- Absolute validity does no longer depend on the distinction between formative and descriptive signs. Popper admits that it depends on the distinction between statements and non-statements. But he emphasizes that whereas the former distinction affects the very central problem, validity, the latter can't affect the decision as to the validity or invalidity. It can only affect the question whether a certain sequence of expressions is an inference (valid or invalid) or no inference at all.
- We can not yet define, for instance, what we mean when we say: "*a* is the negation of *b*". But we do posses the means of defining what we mean when we say:

  "*a* has the same (logical) force as a negation of *b* *whatever the logical <u>form</u> of a and b may be*"

  And this is what Popper gets now ready to do.

THE NOTION OF DEDUCIBILITY

- His program is to characterize the notion of deducibility through a certain system of absolutely valid rules, and then, *without any link to any particular language* ([1947], p. 260), he proceeds to provide definitions of logical compounds just in terms of the metalinguistically characterized deducibility relation.
- At the metalinguistic level, Popper adopts the following symbolic notations:

$$\rightarrow \qquad \leftrightarrow \qquad \& \qquad \vee \qquad (a) \qquad (\exists a)$$

THE NOTION OF DEDUCIBILITY II

- To express the assertion: "From the statements $a_1, \ldots, a_n$, the statement b can be derived" Popper uses the notation

$$a_1, \ldots, a_n / b$$

noting that,

1. the symbols a, b, c, ..., are *variables*, and their values are statements. Since we are at the metalinguistic level, *names of statements* (and not the statements themselves) may be substituted for the variables, which can be described as variable names of statements; and
2. although we may operate with as many premises as we like, we draw only *one* conclusion at a time.

THE NOTION OF DEDUCIBILITY III

- Popper first attempts to determine the notion of deducibility by laying down a few very simple primitive rules for it, called a *Basis*. Basis I consists of
- a Generalised Principle of Reflexivity, referred to by (RG):

$$a_1, \ldots, a_n / a_i \qquad (1 \leq i \leq n)$$

- and a Generalized Principle of Transitivity, referred to by (TG):

$$\frac{\begin{array}{c}(a_1, \ldots, a_n / b_1) \\ \vdots \\ (a_1, \ldots, a_n / b_m)\end{array}}{(b_1, \ldots, b_m / c \rightarrow a_1, \ldots, a_n / c)}$$

THE NOTION OF DEDUCIBILITY IV

- In a preliminary way, it is to be reminded that the introduction of compound statements starts by assuming postulates which assure, for every (pair of) statement(s), say *a* (and *b*), the existence of the corresponding compound statement. The function of postulates, which do not really form a part of Popper's theory of inference, is solely to indicate explicitly that the application of the theory is limited, if we wish to operate with certain compounds, to languages which contain these compounds.
- If *a* and *b* are mutually deducible, we write

  $$a//b$$

  We may also define in a obvious way "//" on the basis of "/":

  (D//) $a//b$ if, and only if, $a/b \quad \& \quad b/a$.

FIRST PROPOSAL

- We focus on "negation": in [1947] the following definition is first given

$$(4.6) \qquad \neg a, b/\neg c \quad \leftrightarrow \quad c, b/a.$$

  It is interesting to note that (4.6) characterizes negation by means of the rule of contraposition in which the left to right direction $\neg a/\neg c \rightarrow c/a$ is an *intuitionistically* invalid form. In other words, (4.6) amounts in effect only to a principle underlying the *classical* theory of "indirect reduction".
- Popper notes that (4.6) is not completely satisfying since *two* negations occur on the left at the same time. Being the one somehow linked to the other, cannot always be eliminated alone.

A BETTER FORMULATION

- Much more satisfying is considered the following definition:

$$\text{(D 5.6)} \qquad a//\neg b \quad \leftrightarrow \quad (a_1)(b_1)(a, a_1/b \rightarrow (a, a_1/b_1 \quad \& \quad a_1, b_1/b)).$$

  Popper notes that the last occurrence of "b_1" could be omitted. It is added only to make obvious the simmetry between the laws of contradiction and excluded middle.

- We think, however, that (5.6) combines in a somewhat cumbersome way a form of Peirce rule – if $\neg b, a_1/b$ then b follows from any pair a_1, b_1 of statements– together with the law of contradiction: from $\neg b, a_1$ follows any b_1.

CLASSICAL AND INTUITIONISTIC NEGATION

- In an interesting way classical negation is compared with the intuitionistic one in the following two definitions, in which occurs just one quantifier, and are therefore not quite suitably related to (D 5.6):

$$a//\neg^c b \quad \leftrightarrow \quad (b_1)(a, b/b_1 \quad \& \quad (a, b_1/b \quad \rightarrow \quad b_1/b)) \tag{D 5.6c}$$

$$a//\neg^i b \quad \leftrightarrow \quad (b_1)(a, b/b_1 \quad \& \quad (b, b_1/a \quad \rightarrow \quad b_1/a)) \tag{D 5.6i}$$

CLASSICAL AND INTUITIONISTIC NEGATION II

- Seen from the point of view of sequent calculus, both rules contain an instance of “ex falso quodlibet”, $(b_1)(\neg^{c,i} b, b / b_1)$, together with an application of the rules of negation (respectively: Peirce’s rule and self-denial) and contraction:

$$
\dfrac{
  \dfrac{\dfrac{\dfrac{\neg^c b, b_1 \vdash b}{\neg^c b, \neg^c b, b_1 \vdash}}{\neg^c b, b_1 \vdash}}{b_1 \vdash \neg^c \neg^c b}
  \qquad
  \dfrac{\dfrac{b \vdash b}{\vdash b, \neg^c b}}{\neg^c \neg^c b \vdash b}
}{b_1 \vdash b}
\qquad\qquad
\dfrac{\dfrac{
  \dfrac{
    \dfrac{b, b_1 \vdash \neg^i b}{\neg^i \neg^i b, b, b_1 \vdash}
    \qquad
    \dfrac{\dfrac{b \vdash b}{\neg^i b, b \vdash}}{b \vdash \neg^i \neg^i b}
  }{b, b, b_1 \vdash}
}{b, b_1 \vdash}}{b_1 \vdash \neg^i b}
$$

EXCLUSIVENESS and EXHAUSTIVENESS

- This provides a first exemplification of the *decomposition* of logical notions we refer to in the title.
- In [1947a, p. 284] the same definition (5.6c) is taken as the starting point for further developing the analysis. In fact, Popper defines the “exclusiveness” (or “contradictoriness”) of a couple of statements [$a \perp b$], and their “exhaustiveness” (or “logical disjunctness”) [$a \top b$] (both notions can be extended to any number $n \geq 2$ of statements).

$$a \perp b \leftrightarrow (c)(d)(c/a \rightarrow (c/b \rightarrow c/d)) \tag{7.5}$$

$$a \top b \leftrightarrow (c)(d)(a/c \rightarrow (b/c \rightarrow d/c)) \tag{7.6}$$

EXCLUSIVENESS and EXHAUSTIVENESS II

- The *Exclusiveness* of a and b expresses the impossibility of their coexistence: if a statement c allows us to infer both a and b, then c allows us to infer any statement d; that is, any c capable to separately *infer* two exclusive sentences plays the role of "falsum": $c \equiv \bot$. A pair of exclusive sentences is the *weakest* "sentence" capable to infer any other sentence of the language.
- The *Exhaustiveness* of a and b expresses the fact that a and b fill any possibility: if a statement c can be inferred from both a and b, then c can be inferred from any statement d; that is, any statement which can be separately *obtained* from two exhaustive statements plays the role of "verum": $c \equiv \top$. A pair of exhaustive sentences is the *strongest* "sentence" capable to be inferred from any other sentence of the language.

COMPLEMENTARITY

- By means of definitions (7.5) and (7.6), Popper defines the notion of complementarity of statements a and b:

$$(7.7) \qquad a// \text{ the complement of } b$$

$$\leftrightarrow$$

$$((a \quad \perp \quad b) \quad \& \quad (a \quad \top \quad b))$$

If a and b are exclusive as well as exhaustive, then

$$a// \text{ the complement of } b$$

This means that if a is the complement of b then a and b cannot coexist whereas they cover any possibility. In a sense, “complement” and “negation” are equivalent notions:

$$[a//\neg b \quad \& \quad c// \text{ the complement of } b] \rightarrow (a//c)$$

COMPLEMENTARITY II

- The two components of the notion of "complementarity" (and, through the equivalence, of the notion of "negation") focus on different features of the group of the "identity rules": exclusiveness looks at the "$\ldots \vdash$" perspective, whereas exhaustiveness looks at the "$\vdash \ldots$" direction.
- This analysis is already available in definition (7.2) of [1947a]

$$(7.2) \qquad a//\text{the negation of } b \text{ if, and only if,}$$
$$(c)(a, b/c \quad \& \quad (a, c/b \quad \rightarrow \quad c/b)).$$

according to which "negation" has two components: the former exhibits (a variant of) the "$\ldots \vdash$" perspective, the latter expresses the content of the Peirce's rule, and is a variant of the "$\vdash \ldots$" perspective.

INTUITIONISTIC NEGATION

- Deepening the previous decomposition, in [1948a] Popper reminds that

  > *The intuitionistic negation of b is the* weakest *of those statements which are* strong *enough to contradict b (my emphasis)*

  meaning that, together with b, it is capable to infer any c.
- Thus, a is equivalent to the intuitionistic negation of b, let's say $a//\neg^i b$, if and only if

$$(c)(c/a \quad \leftrightarrow \quad (d)(e)((d/c \quad \& \quad d/b) \quad \rightarrow \quad d/e).$$

CLASSICAL NEGATION ?

- Popper comments in [1948a] stressing that intuitionistic negation is characterized by contradictoriness, or exclusiveness, alone. This fact could induce the idea that an analogous link could exist between classical negation and complementarity, or exhaustiveness, *again* alone.
- However, this is an idea we must abandon: it would mean that a is equivalent to the classical negation of b, let's say $a//\neg^c b$, if and only if

  $$(c)(c/a \leftrightarrow (d)(e)((a/d \quad \& \quad b/d \quad \rightarrow \quad e/d))$$

  where we have dropped the part $(c)(a, b/c)$, the "principle of contradiction".

MINIMUM DEFINABLE NEGATION

- Thus, the idea to be adandoned is that "excluded middle" *alone* could be enough to characterize classical negation, meaning that any c which can be inferred from both a statement and its classical negation, can be inferred from any statement. In other words, this is evidence for the mutual independence of the two components of (classical) negation.
- Pursuing this idea, Popper gets a different (from both classical and intuitionistic) notion of negation, say $\neg^m b$, which is called the "minimum definable negation of b":

$$\text{(D 4.2)} \qquad a//\neg^m b \quad \leftrightarrow \quad (c)(a/c \quad \leftrightarrow \quad (d)(e)((b/e \quad \& \quad c/e) \rightarrow d/e)$$

DUAL INTUITIONISTIC LOGIC

- Popper comments on (D 4.2) saying that $\neg^m b$ is the strongest of those statements which are weak enough to be complements of *b*. The right hand side of (D 4.2) means that a statement *c* can be inferred from $\neg^m b$ iff it is complementary to *b*.
- Deconstructing and reassembling logical notions, specifically *negation*, Popper has been able to draw attention to a "new" logic, which is now called *dual intuitionism*, and which is characterized by the "minimum definable negation", and which can have at most one formula on the left of the sequent arrow.

OPPONENT

- Continuing to dwell upon the various notions, Popper notes that besides taking care to use different names, there is no need to make sure that our system of definitions, and hence the language object whose inferential relations we are dealing with, is consistent.
- This is most probably a teaching which is part of A. Tarski's legacy. I mean, of his claim regarding the inconsistency of natural languages. Popper makes the example of the notion of "opponent" (we refer to [1947a]):

$$(7.8) \qquad a// \text{ opp } (b) \text{ if, and only if, } (c)(b/a \quad \& \quad a/c).$$

TONK

- Popper emphasizes that, as a consequence of definition (7.8), every language which has a sign for “opponent of *b*” – analogous to the sign for “negation of *b*” – will be inconsistent. But this need not lead us to abandon (7.8); it only means that no consistent language will have a sign for “opponent of *b*”.
- Popper’s definition of “opponent of *b*” seems to contain already the idea of Prior’s connective “tonk”, and in fact Prior cites Popper’s paper (even though just with regard to the clarification of the notion of “analytically valid inference” provided by Popper in [1947a]).

TONK II

- However, Popper doesn't feel obliged to raise barriers around the notion of "opponent". We can say that "opponent" is a(n *ante litteram*) generalization of "tonk"; in the sense that we get "tonk" when the (c) in the definition of "opp b" is particularized to b. In this way, in fact, we have that "$a \quad tonk \quad b$" is the opponent of a; in fact, by the same rules governing "tonk" it holds:

$$(a/a \quad tonk \quad b \quad \& \quad a \quad tonk \quad b/b).$$

TONK III

- Reflecting on *opp*(b), and with respect to the *inferential definitions* provided by Popper for the logical compounds of a given object language $\mathcal{L}$, it is reasonable to wonder if any set of rules gives rise to a definition of a logical constant or not.
- The question is very near to the one people working in proof-theoretic semantics had to face after “tonk” came to the fore: is every “inferential definition” to be allowed? Popper’s answer seems to be “yes”, and this would teach us that his system seems not include any harmony requirement, since the transitivity rule used by Belnap to overcome “tonk” is part of Basis I.

UNIQUENESS

- As D. Binder and T. Piecha say in [2017],

> *there is one condition that Popper seems to consider to be essential for any definition of a logical constant, namely uniqueness:[. . .] fully characterizing rules are exactly those rules that satisfy uniqueness.[. . .] It is the existence of fully characterizing rules that distinguishes logical constants from non-logical constants, and it is this criterion of logicality that leads Popper to reject, for example, minimal negation as a logical constant. (pp. 167-168)*

THANKS

FOR YOUR ATTENTION

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