Polish Topologies for Graph Products of Groups

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The Beginning of the Story

Question (Evans)

Can an uncountable free group be the automorphism group of a countable structure?

Answer (Shelah)

No uncountable free group can be the group of automorphisms of a countable structure.

Polish Groups

Definition

A Polish group is a topological group whose topology is separable and completely metrizable.

Fact

Groups of automophisms of countable structures are Polish groups (such groups are called non-archimedean Polish groups).

Answer (Shelah)

No uncountable free group admits a Polish group topology.

The Completeness Lemma for Polish Groups

In order to settle the above Shelah proved what he called the Completeness Lemma for Polish Groups.

This is a technical result stating that if G is a Polish group, then for every sequence $\bar{d} = (d_n : n < \omega) \in G^\omega$ converging to the identity e_G, many countable sets of equations with parameters from $\bar{d}$ are solvable in G.

Right-Angled Artin Groups

Definition

Given a graph $\Gamma = (E, V)$, the associated right-angled Artin group (a.k.a RAAG) $A(\Gamma)$ is the group with presentation:

$$\Omega(\Gamma) = \langle V \mid ab = ba : aEb \rangle.$$

If in the presentation $\Omega(\Gamma)$ we ask in addition that all the generators have order 2, then we speak of right-angled Coxeter groups (a.k.a RACG) $C(\Gamma)$.

No Uncountable Polish group can be a RAAG

Theorem (P. & Shelah)

Let $G = (G, d)$ be an uncountable Polish group and A a group admitting a system of generators whose associated length function satisfies the following conditions:

(i) *if $0 < k < \omega$, then $lg(x) \leqslant lg(x^k)$;*

(ii) *if $lg(y) < k < \omega$ and $x^k = y$, then $x = e$.*

Then G is not isomorphic to A, in fact there exists a subgroup G^* of G of size $\mathfrak{b}$ (the bounding number) such that G^* is not embeddable in A.

Corollary (P. & Shelah)

No uncountable Polish group can be a right-angled Artin group.

What about right-angled Coxeter groups?

The structure M with ω many disjoint unary predicates of size 2 is such that $Aut(M) = (\mathbb{Z}_2)^\omega$, i.e. $Aut(M)$ is the right-angled Coxeter group on the complete graph $K_\mathfrak{c}$.

Question

Which right-angled Coxeter groups admit a Polish group topology?

Graph Products of Cyclic Groups

Definition

Let $\Gamma = (V, E)$ be a graph and

$$\mathfrak{p} : V \to \{p^n : p \text{ prime and } 1 \leqslant n\} \cup \{\infty\}$$

a graph coloring. We define a group $G(\Gamma, \mathfrak{p})$ with the following presentation:

$$\langle V \mid a^{\mathfrak{p}(a)} = 1,\ bc = cb : \mathfrak{p}(a) \neq \infty \text{ and } bEc \rangle.$$

A Characterization

Theorem (P. & Shelah)

Let $G = G(\Gamma, \mathfrak{p})$. Then G admits a Polish group topology if only if $(\Gamma, \mathfrak{p})$ satisfies the following four conditions:

(a) *there exists a countable $A \subseteq \Gamma$ such that for every $a \in \Gamma$ and $a \neq b \in \Gamma - A$, a is adjacent to b;*

(b) *there are only finitely many colors c such that the set of vertices of color c is uncountable;*

(c) *there are only countably many vertices of color ∞;*

(d) *if there are uncountably many vertices of color c, then the set of vertices of color c has the size of the continuum.*

Furthermore, if $(\Gamma, \mathfrak{p})$ satisfies conditions (a)-(d) above, then G can be realized as the group of automorphisms of a countable structure.

In Plain Words

Theorem (P. & Shelah)

The only graph products of cyclic groups $G(\Gamma, \mathfrak{p})$ *admitting a Polish group topology are the direct sums* $G_1 \oplus G_2$ *with* G_1 *a countable graph product of cyclic groups and* G_2 *a direct sum of finitely many continuum sized vector spaces over a finite field.*

Embeddability of Graph Products into Polish groups

Fact

The free group on continuum many generators is embeddable into the automorphism group of the random graph.

Question

Which graph products of cyclic groups $G(\Gamma, \mathfrak{p})$ *are embeddable into a Polish group?*

Another Characterization

Theorem (P. & Shelah)

Let $G = G(\Gamma, \mathfrak{p})$, then the following are equivalent:

(a) *there is a metric on Γ which induces a separable topology in which E_Γ is closed;*

(b) *G is embeddable into a Polish group;*

(c) *G is embeddable into a non-Archimedean Polish group.*

Even More...

Theorem (P. & Shelah)

Let $\Gamma = (\omega^\omega, E)$ *be a graph and*

$$\mathfrak{p} : V \to \{p^n : p \text{ prime}, n \geqslant 1\} \cup \{\infty\}$$

a graph coloring. Suppose further that E *is closed in the Baire space* ω^ω*, and that* $\mathfrak{p}(\eta)$ *depends only on* $\eta(0)$*. Then* $G = G(\Gamma, \mathfrak{p})$ *admits a left-invariant separable group ultrametric extending the standard metric on the Baire space.*

The Last Level of Generality

Definition

Let $\Gamma = (V, E)$ be a graph and $\{G_a : a \in \Gamma\}$ a set of non-trivial groups each presented with its multiplication table presentation and such that for $a \neq b \in \Gamma$ we have $e_{G_a} = e = e_{G_b}$ and $G_a \cap G_b = \{e\}$. We define the graph product of the groups $\{G_a : a \in \Gamma\}$ over Γ, *denoted $G(\Gamma, G_a)$, via the following presentation:*

$$\textit{generators: } \bigcup_{a \in V} \{g : g \in G_a\},$$

$$\textit{relations: } \bigcup_{a \in V} \{\textit{the relations for } G_a\}$$

$$\cup \bigcup_{\{a,b\} \in E} \{gg' = g'g : g \in G_a \textit{ and } g' \in G_b\}.$$

Some Notation

Notation

(1) *We denote by* $\mathbb{Q} = G^*_\infty$ *the rational numbers and by* $\mathbb{Z}_{p^k} = G^*_{(p,k)}$ *the finite cyclic group of order* p^k *(for* p *a prime and* $k \geqslant 1$*).*

(2) *We let* $S_* = \{(p,k) : p \text{ prime and } k \geqslant 1\} \cup \{\infty\}$*.*

(3) *For* $s \in S_*$ *and* λ *a cardinal, we let* $G^*_{s,\lambda}$ *be the direct sum of* λ *copies of* G^*_s*.*

The First Venue

Theorem (P. & Shelah)

Let $G = G(\Gamma, G_a)$ and suppose that G admits a Polish group topology. Then for some countable $A \subseteq \Gamma$ and $1 \leqslant n < \omega$ we have:

(a) *for every $a \in \Gamma$ and $a \neq b \in \Gamma - A$, a is adjacent to b;*

(b) *if $a \in \Gamma - A$, then $G_a = \bigoplus\{G^*_{s,\lambda_{a,s}} : s \in S_*\}$;*

(c) *if $\lambda_{a,(p,k)} > 0$, then $p^k \mid n$;*

(d) *if in addition $A = \emptyset$, then for every $s \in S_*$ we have that $\sum\{\lambda_{a,s} : a \in \Gamma\}$ is either $\leqslant \aleph_0$ or $2^{\aleph_0}$.*

The Third Venue

Corollary (P. & Shelah)

Let $G = G(\Gamma, G_a)$ with all the G_a countable. Then G admits a Polish group topology if only if G admits a non-Archimedean Polish group topology if and only if there exist a countable $A \subseteq \Gamma$ and $1 \leqslant n < \omega$ such that:

(a) *for every $a \in \Gamma$ and $a \neq b \in \Gamma - A$, a is adjacent to b;*

(b) *if $a \in \Gamma - A$, then $G_a = \bigoplus \{ G^*_{s, \lambda_{a,s}} : s \in S_* \}$;*

(c) *if $\lambda_{a,(p,k)} > 0$, then $p^k \mid n$;*

(d) *for every $s \in S_*$, $\sum \{ \lambda_{a,s} : a \in \Gamma - A \}$ is either $\leqslant \aleph_0$ or $2^{\aleph_0}$.*

The Third Venue (Cont.)

Corollary (P. & Shelah)

Let G be an abelian group which is a direct sum of countable groups, then G admits a Polish group topology if only if G admits a non-Archimedean Polish group topology if and only if there exists a countable $H \leqslant G$ and $1 \leqslant n < \omega$ such that:

$$G = H \oplus \bigoplus_{\alpha<\lambda_\infty} \mathbb{Q} \oplus \bigoplus_{p^k \mid n} \bigoplus_{\alpha<\lambda_{(p,k)}} \mathbb{Z}_{p^k},$$

with λ_∞ and $\lambda_{(p,k)} \leqslant \aleph_0$ or $2^{\aleph_0}$.

Corollary (P. & Shelah, and independently Slutsky)

If G is an uncountable group admitting a Polish group topology, then G can not be expressed as a non-trivial free product.

A Conjecture

Conjecture (Polish Direct Summand Conjecture)

Let G be a group admitting a Polish group topology.

(1) *If G has a direct summand isomorphic to $G^*_{s,\lambda}$, for some $\aleph_0 < \lambda \leqslant 2^{\aleph_0}$ and $s \in S_*$, then it has one of cardinality $2^{\aleph_0}$.*

(2) *If $G = G_1 \oplus G_2$ and $G_2 = \bigoplus\{G^*_{s,\lambda_s} : s \in S_*\}$, then for some G'_1, G'_2 we have:*
   - (i) $G_1 = G'_1 \oplus G'_2$;
   - (ii) G'_1 *admits a Polish group topology;*
   - (iii) $G'_2 = \bigoplus\{G^*_{s,\lambda'_s} : s \in S_*\}$.

(3) *If $G = G_1 \oplus G_2$, then for some G'_1, G'_2 we have:*
   - (i) $G_1 = G'_1 \oplus G'_2$;
   - (ii) G'_1 *admits a Polish group topology;*
   - (iii) $G'_2 = \bigoplus\{G^*_{s,\lambda_s} : s \in S_*\}$.