Typoids in Martin-Löf's Intensional Type Theory

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Special features of ITT

1. It is based on various kinds of inductive definitions
2. Logic is built-in (logic-free, D. Scott: Constructive validity, 1970)
3. Distinction between "Propositions and Judgements"
4. Equality is specific to each set (Bishop), but in a "global" way (Martin-Löf)
5. Propositional equality of a type is the least reflexive relation on it
6. Decidability of type-checking
7. It is a programming language. Coquand et.al.: "this is a major compeling aspect of ITT compared to non-constructive foundations such as set theory".

The canonicity property of ITT

Canonicity Property (CP): Every closed term of type **N** is simplified (reduced) to a numeral.

Let $A \in \{$function-extensionality, univalence axiom, higher inductive types, PEM, Brouwer's continuity axiom, bar induction$\}$.

ITT does not prove A

ITT $+ A$ consistent, but loses canonicity

Coquand et.al.: "ITT still has a proof assistant, but the proof language ceases to be a programming language"

Coquand et.al (2013): $S = \text{ITT} + (c_n : \neg A_n)_n$ has the CP, if S doesn't inhabit the empty type with a closed term.

CP is open in HoTT = ITT + UA + HITs

Huber (2016): Cubical type theory has CP (looks quite different than ITT).

$\mathrm{Form}_{x=_A y}$: If $x : A$ and $y : A$, the **equality type** $x =_A y : \mathcal{U}$.
$\mathrm{Intro}_{x=_A x}$:

$$\mathtt{refl}_A : \prod_{x:A} x =_A x.$$

$\mathrm{Ind}_{=_A}$: If

$$C : \prod_{x,y:A} \prod_{p:x=_A y} \mathcal{U}$$

is a dependent family of types in $\mathcal{U}$, and if

$$c : \prod_{x:A} C(x, x, \mathtt{refl}_x)$$

is a dependent function, there is a dependent function

$$F : \prod_{x,y:A} \prod_{p:x=_A y} C(x, y, p)$$

such that

$$F(x, x, \mathtt{refl}_x) \equiv c(x).$$

This is the inductive definition of the type family $=_A : A \to A \to \mathcal{U}$ with two indices in A and with constructor

$$\frac{x : A}{\mathtt{refl}_x : x =_A x}$$

The type $x =_A y$ is NOT defined inductively, but the type family is.

$$J : \prod_{A:\mathcal{U}} \prod_{C:\prod_{x,y:A}\prod_{p:x=_A y}\mathcal{U}} \prod_{c:\prod_{x:A} C(x,x,\mathtt{refl}_x)} \prod_{\prod_{x,y:A}\prod_{p:x=_A y}} C(x,y,p)$$

$$J(A, C, c, x, x, \mathtt{refl}_x) \equiv c(x)$$

$$\mathtt{LeastRefl} : \prod_{A:\mathcal{U}} \prod_{R:A\to A\to\mathcal{U}} \prod_{r:\prod_{x:A} R(x,x)} \prod_{x,y:A} \prod_{p:x=_A y} R(x,y)$$

$$\mathtt{LeastRefl}(A, R, r, x, x, \mathtt{refl}_x) \equiv r(x)$$

$$\texttt{Transport} : \prod_{A:\mathcal{U}} \prod_{P:A\to\mathcal{U}} \prod_{x,y:A} \prod_{p:x=_A y} P(x) \to P(y)$$

$$\texttt{Transport}(A, P, x, x, \texttt{refl}_x) \equiv \mathrm{id}_{P(x)}$$

$$p_*^P$$

$$\texttt{Application} : \prod_{A,B:\mathcal{U}} \prod_{f:A\to B} \prod_{x,y:A} \prod_{p:x=_A y} f(x) =_B f(y)$$

$$\texttt{Application}(A, B, f, x, x, \texttt{refl}_x) \equiv \texttt{refl}_{f(x)}$$

$$\texttt{ap}_f(x, y)$$

Setoids

$$\mathtt{isProp}(B) \equiv \prod_{x,y:B} (x =_B y)$$

$$\sim_A : A \to A \to \mathcal{U}$$

$$\mathtt{isProp}(x \sim_A y)$$

$$\prod_{x,y:A} \prod_{e:x \sim_A y} f(x) \sim_B f(y).$$

$$(x, y) \simeq_{A \times B} (x', y') \equiv (x \simeq_A x') \times (y \simeq_B y')$$

$$B^A \equiv \sum_{f:A \to B} \prod_{x,y:A} (x \sim_A y \to f(x) \sim_B f(y))$$

$$(f, u) \sim_{B^A} (g, w) \equiv \prod_{x:A} (f(x) = g(x))$$

Setoids and setoid functions form a cartesian closed category.
We can realize function extensionality in ITT via the setoid B^A.

Equivalence of types

$$f \sim g :\equiv \prod_{x:A} (f(x) =_B g(x)).$$

$$A \simeq_{\mathcal{U}} B :\equiv \sum_{f:A\to B} \mathtt{isequiv}(f),$$

$$\mathtt{isequiv}(f) :\equiv \left(\sum_{g:B\to A} (f \circ g) \sim \mathrm{id}_B) \right) \times \left(\sum_{h:B\to A} (h \circ f) \sim \mathrm{id}_A) \right).$$

$$\mathrm{qinv}(f) :\equiv \sum_{g:B\to A} [(f \circ g \sim \mathrm{id}_B) \times (g \circ f \sim \mathrm{id}_A)].$$

$$\mathrm{qinv}(f) \leftrightarrow \mathtt{isequiv}(f).$$

$$\mathtt{eqv}_1, \mathtt{eqv}_2 : \mathtt{isequiv}(f) \Rightarrow \mathtt{eqv}_1 = \mathtt{eqv}_2.$$

Function extensionality

$$\begin{aligned} \text{intro}: &\quad \texttt{funext} : f \sim g \to f = g \\ \text{elim}: &\quad \texttt{happly} : f = g \to f \sim g \\ \text{propcomprule}: &\quad \texttt{happly}(\texttt{funext}(H), x) = H(x) \\ \text{propuniqrule}: &\quad \texttt{funext}(\texttt{happly}(p)) = p \end{aligned}$$

$$\texttt{funext}(\texttt{eq}_f) = \texttt{refl}_f,$$

$$\texttt{eq}_f(x) \equiv \texttt{refl}_{f(x)}$$

$$\texttt{funext}(\texttt{happly}(p)^{-1}) = p^{-1},$$

$$\texttt{happly}(p)^{-1})(x) \equiv \texttt{happly}(p, x)^{-1}$$

$$\texttt{funext}(\texttt{happly}(p * q)) = \texttt{funext}(\texttt{happly}(p)) * \texttt{funext}(\texttt{happly}(q)),$$

$$\texttt{happly}(p * q, x) \equiv \texttt{happly}(p, x) * \texttt{happly}(q, x).$$

Univalence axiom

$$\begin{aligned}
\text{intro}: &\quad \mathtt{ua} : A \simeq_{\mathcal{U}} B \to A =_{\mathcal{U}} B \\
\text{elim}: &\quad \mathtt{IdtoEqv} : A =_{\mathcal{U}} B \to A \simeq_{\mathcal{U}} B \\
\text{propcomprule}: &\quad \mathtt{IdtoEqv}(\mathtt{ua}(f), x) = f(x) \\
\text{propuniqrule}: &\quad \mathtt{ua}(\mathtt{IdtoEqv}(p)) = p
\end{aligned}$$

$$\mathtt{ua}(\mathrm{id}_A) = \mathtt{refl}_A$$

$$\mathtt{ua}(g \circ f) = \mathtt{ua}(f) * \mathtt{ua}(g)$$

$$\mathtt{ua}(f)^{-1} = \mathtt{ua}(f)^{-1}$$

If $A : \mathcal{U}, a : A, R : A \to \mathcal{U}$ and $r : R(a)$, the structure (A, a, R, r) is called an **identity system at** a, if for every

$$D : \prod_{x:A} \prod_{p:R(x)} \mathcal{U}, \qquad d : D(a, r)$$

there is

$$F : \prod_{x:A} \prod_{p:R(x)} D(x, p)$$

such that

$$F(a, r) = d.$$

Theorem (5.8.2 in HoTT-book)

(A, a, R, r) is an identity system at a iff for every $x : A$ the function

$$\mathtt{uf} : (a =_A x) \to R(x)$$

$$\mathtt{uf}(p) \equiv p_*^R(r)$$

$$p_*^R : R(a) \to R(x)$$

is an equivalence.

A **typoid** is a structure $\mathcal{A} \equiv (A, \simeq_{\mathcal{A}}, \mathrm{eqv}_{\mathcal{A}}, *_{\mathcal{A}}, ^{-1_{\mathcal{A}}}, \cong_{\mathcal{A}})$, s.t.

$$\mathrm{eqv}_{\mathcal{A}} : \prod_{x:A} (x \simeq_{\mathcal{A}} x),$$

$$*_{\mathcal{A}} : \prod_{x,y,z:A} \prod_{e:x \simeq_{\mathcal{A}} y} \prod_{d:y \simeq_{\mathcal{A}} z} x \simeq_{\mathcal{A}} z,$$

$$^{-1_{\mathcal{A}}} : \prod_{x,y:A} \prod_{e:x \simeq_{\mathcal{A}} y} y \simeq_{\mathcal{A}} x$$

(i) $(\mathrm{eqv}_x *_{\mathcal{A}} e) \cong_{\mathcal{A}} e$ and $(e *_{\mathcal{A}} \mathrm{eqv}_y) \cong_{\mathcal{A}} e$.

(ii) $(e *_{\mathcal{A}} e^{-1_{\mathcal{A}}}) \cong_{\mathcal{A}} \mathrm{eqv}_x$ and $(e^{-1_{\mathcal{A}}} *_{\mathcal{A}} e) \cong_{\mathcal{A}} \mathrm{eqv}_y$.

(iii) $(e_1 *_{\mathcal{A}} e_2) *_{\mathcal{A}} e_3 \cong_{\mathcal{A}} e_1 *_{\mathcal{A}} (e_2 *_{\mathcal{A}} e_3)$.

(iv) $e_1 \cong_{\mathcal{A}} d_1 \to e_2 \cong_{\mathcal{A}} d_2 \to (e_1 *_{\mathcal{A}} e_2) \cong_{\mathcal{A}} (d_1 *_{\mathcal{A}} d_2)$.

$$\texttt{Typoid}(\mathcal{A}) \equiv \sum_{A:\mathcal{U}} \sum_{\simeq_{\mathcal{A}}:\prod_{x,y:A}\mathcal{U}} \sum_{\texttt{eqv}_{\mathcal{A}}:\prod_{x:A}(x \simeq_{\mathcal{A}} x)}$$

$$\sum_{*_{\mathcal{A}}:\prod_{x,y,z:A}\prod_{e:x\simeq_{\mathcal{A}}y}\prod_{d:y\simeq_{\mathcal{A}}z} x\simeq_{\mathcal{A}}z} \quad \sum_{{}^{-1_{\mathcal{A}}}:\prod_{x,y:A}\prod_{e:x\simeq_{\mathcal{A}}y} y\simeq_{\mathcal{A}}x}$$

$$\sum_{\cong_{\mathcal{A}}:\prod_{x,y:A}\prod_{e,e':x\simeq_{\mathcal{A}}y}} \Big((i) \times (ii) \times (iii) \times (iv)\Big).$$

Actually, this could be seen as a 2-**typoid**.

$$(\texttt{eqv}_x)^{-1_{\mathcal{A}}} \cong_{\mathcal{A}} \texttt{eqv}_x$$

$$(e^{-1_{\mathcal{A}}})^{-1_{\mathcal{A}}} \cong_{\mathcal{A}} e$$

$$e \cong_{\mathcal{A}} d \to e^{-1} \cong_{\mathcal{A}} d^{-1}$$

Using fundamental properties of equality $p =_{x =_A y} q$, of concatenation $p * q$ and inversion p^{-1} of paths it is easy to see that

$$\mathcal{A}_0 \equiv (A, =_A, \mathtt{refl}_A, *, ^{-1}, \cong_{\mathcal{A}_0}),$$

where $\cong_{\mathcal{A}_0} : \prod_{x,y:A} \prod_{e,e' : x =_A y} \mathcal{U}$ is defined by

$$\cong_{\mathcal{A}_0} (x, y, e, e') \equiv (e =_{x =_A y} e'),$$

for every $x, y : A$ and $e, e' : x =_A y$, is a typoid. We call $\mathcal{A}_0$ the **equality** typoid, and its typoid structure the **equality** typoid structure on A.

$$(A \to B, \simeq_{A\to B}, \mathtt{eqv}_{A\to B}, *_{A\to B}, {}^{-1_{A\to B}}, \cong_{A\to B})$$

is the **typoid of functions**, where

$$f \simeq_{A\to B} g \equiv \prod_{x:A} f(x) =_B g(x),$$

while if $H, H' : f \simeq_{A\to B} g$ and $G : g \simeq_{A\to B} h$, we define

$$H *_{A\to B} G \equiv \lambda(x : A).(H(x) * G(x)),$$

$$H^{-1_{A\to B}} \equiv \lambda(x : A).(H(x))^{-1},$$

$$\mathtt{eqv}_f \equiv \lambda(x : A).\mathtt{refl}_{f(x)},$$

$$H \cong_{A\to B} H' \equiv \prod_{x:A} H(x) =_{(f(x)=_B g(x))} H'(x).$$

$$\mathrm{Uni} \equiv (\mathcal{U}, \simeq_{\mathcal{U}}, \mathtt{eqv}_{\mathcal{U}}, *_{\mathcal{U}}, {}^{-1_{\mathcal{U}}}, \cong_{\mathcal{U}})$$

is the **universal typoid**, where

$$A \simeq_{\mathcal{U}} B \equiv \sum_{f : A \to B} \mathtt{isequiv}(f),$$

while if $(f, u), (f', u') : A \simeq_{\mathcal{U}} B$ and $(g, v) : B \simeq_{\mathcal{U}} C$, we define

$$(f, u) *_{\mathcal{U}} (g, v) \equiv (g \circ f, w),$$

$$(f, u)^{-1_{\mathcal{U}}} \equiv (f^{-1}, u^{-1}),$$

$$\mathtt{eqv}_A \equiv (\mathrm{id}_A, i),$$

$$(f, u) \cong_{\mathcal{U}} (f', u') \equiv \prod_{x : A} f(x) =_B f'(x),$$

where $w : \mathtt{isequiv}(g \circ f), u^{-1} : \mathtt{isequiv}(f^{-1})$ and
$i : \mathtt{isequiv}(\mathrm{id}_A)$. Note that the definition of $(f, u) \cong_{\mathcal{U}} (f', u')$ is based on the fact that all terms of type $\mathtt{isequiv}(f)$ are equal.

If $\mathcal{A}, \mathcal{B}$ are typoids, $f : A \to B$ is a **typoid function**, if there are

$$\Phi_f : \prod_{x,y:A} \prod_{e:x \simeq_{\mathcal{A}} y} f(x) \simeq_{\mathcal{B}} f(y),$$

$$\Phi_f^2 : \prod_{x,y:A} \prod_{e,d:x \simeq_{\mathcal{A}} y} \prod_{i:e \cong_{\mathcal{A}} d} \Phi_f(x,y,e) \cong_{\mathcal{B}} \Phi_f(x,y,d),$$

an 1-**associate** of f and a 2-**associate** of f w.r.t. Φ_f, s.t.

(i) $\Phi_f(x,x,\mathrm{eqv}_x) \cong_{\mathcal{B}} \mathrm{eqv}_{f(x)}$,
(ii) $\Phi_f(x,z,e_1 *_A e_2) \cong_{\mathcal{B}} \Phi_f(x,y,e_1) *_{\mathcal{B}} \Phi_f(y,z,e_2)$.

If $\Phi_f(x,x,\mathrm{eqv}_x) \equiv \mathrm{eqv}_{f(x)}$, f is **strict** w.r.t. Φ_f.

$$\Phi_f(y,x,e^{-1_{\mathcal{A}}}) \cong_{\mathcal{B}} [\Phi_f(x,y,e)]^{-1_{\mathcal{B}}}$$

1. If $\mathcal{A}_0, \mathcal{B}_0$ are equality typoids and $f : A \to B$, then f is a strict typoid function with respect to its 1-associate ap_f and the 2-associate ap^2_f of f.

2. If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are typoids and $f : A \to B, g : B \to C$ are typoid functions with associates Φ_f, Φ^2_f and Φ_g, Φ^2_g, respectively, then $g \circ f : A \to C$ is a typoid function with associates

$$\Phi_{g\circ f} : \prod_{x,y:A} \prod_{e:x\simeq_{\mathcal{A}} y} g(f(x)) \simeq_C g(f(y)),$$

$$\Phi^2_{g\circ f} : \prod_{x,y:A} \prod_{e,d:x\simeq_{\mathcal{A}} y} \prod_{i:e\cong_{\mathcal{A}} d} \Phi_{g\circ f}(x,y,e) \cong_C \Phi_{g\circ f}(x,y,d),$$

$$\Phi_{g\circ f}(x,y,e) \equiv \Phi_g\Big(f(x), f(y), \Phi_f(x,y,e)\Big),$$

$$\Phi^2_{g\circ f}(x,y,e,d,i) \equiv \Phi^2_g\Big(f(x), f(y), \Phi_f(x,y,e),$$

$$\Phi_f(x,y,d), \Phi^2_f(x,y,e,d,i)\Big).$$

If f, g are strict w.r.t. Φ_f, Φ_g, $g \circ f$ is strict w.r.t. $\Phi_{g\circ f}$.

Proposition

If $\mathcal{A}$ is a typoid, the identity function $\mathrm{id}_A : A \to A$ is a typoid function from $\mathcal{A}_0$ to $\mathcal{A}$, which is strict with respect to its 1-associate

$$\texttt{idtoEqv}_{\mathcal{A}} : \prod_{x,y:A} \prod_{p:x=_A y} x \simeq_{\mathcal{A}} y,$$

$$\texttt{idtoEqv}_{\mathcal{A}}(x, y, p) \equiv p_*^{P_x}(\texttt{eqv}_x),$$

where $P_x : A \to \mathcal{U}$ is defined by $P_x(z) \equiv x \simeq_{\mathcal{A}} z$, for every $z : A$.

Note that $p_*^{P_x} : P_x(x) \to P_x(y)$ i.e., $p_*^{P_x} : x \simeq_{\mathcal{A}} x \to x \simeq_{\mathcal{A}} y$. We use path-induction to define $\texttt{idtoEqv}^2_{\mathcal{A}}$.

Proposition

If $\mathcal{A}, \mathcal{B}$ are typoids, then the structure

$$\mathcal{A} \times \mathcal{B} \equiv (A \times B, \simeq_{\mathcal{A}\times\mathcal{B}}, \mathtt{eqv}_{\mathcal{A}\times\mathcal{B}}, *_{\mathcal{A}\times\mathcal{B}}, {}^{-1_{\mathcal{A}\times\mathcal{B}}}, \cong_{\mathcal{A}\times\mathcal{B}})$$

is a typoid, where for every $z, w, u : A \times B$ and $e, e' : z =_{A\times B} w$, $d : w =_{A\times B} u$ we define

$$\mathtt{eqv}_z \equiv T(z, z, \mathtt{eqv}_{\mathtt{pr}_1(z)}, \mathtt{eqv}_{\mathtt{pr}_2(z)}),$$

$$e *_{\mathcal{A}\times\mathcal{B}} d \equiv T(z, u, e_1 *_{\mathcal{A}} d_1, e_2 *_{\mathcal{B}} d_2),$$

$$e^{-1_{\mathcal{A}\times\mathcal{B}}} \equiv T(w, z, e_1^{-1_{\mathcal{A}}}, e_2^{-1_{\mathcal{B}}}),$$

$$e \cong_{\mathcal{A}\times\mathcal{B}} e' \equiv (e_1 \cong_{\mathcal{A}} e_1') \times (e_2 \cong_{\mathcal{B}} e_2').$$

Corollary

If $\mathcal{A}, \mathcal{B}$ are typoids, then $\mathtt{pr}_1, \mathtt{pr}_2$ are typoid functions.

Our first motivation for the study of typoids

Definition

A typoid $\mathcal{A}$ is called **univalent**, if there are dependent functions

$$\mathtt{Ua}_{\mathcal{A}} : \prod_{x,y:A} \prod_{e:x \simeq_{\mathcal{A}} y} x =_A y,$$

$$\mathtt{Ua}^2_{\mathcal{A}} : \prod_{x,y:A} \prod_{e,d:x \simeq_{\mathcal{A}} y} \prod_{i:e \cong_{\mathcal{A}} d} \mathtt{Ua}_{\mathcal{A}}(x,y,e) = \mathtt{Ua}_{\mathcal{A}}(x,y,d)$$

such that for every $x, y : A$, $p : x =_A y$ and $e : x \simeq_{\mathcal{A}} y$ we have that

$$\mathtt{Ua}_{\mathcal{A}}(x,y,\mathtt{IdtoEqv}_{\mathcal{A}}(x,y,p)) = p,$$

$$\mathtt{IdtoEqv}_{\mathcal{A}}(x,y,\mathtt{Ua}_{\mathcal{A}}(x,y,e)) \cong_{\mathcal{A}} e,$$

where $\mathtt{IdtoEqv}_{\mathcal{A}}$ is an 1-associate of id_A (from $\mathcal{A}_0$ to $\mathcal{A}$) w.r.t. which id_A is strict. We call a univalent typoid **strictly** univalent, if

$$\mathtt{Ua}_{\mathcal{A}}(x,x,\mathtt{eqv}_x) \equiv \mathtt{refl}_x.$$

1. The **equality** typoid $\mathcal{A}_0$ is strictly univalent, if we consider $\texttt{IdtoEqv}_{\mathcal{A}}(x, y, p) \equiv p \equiv \texttt{Ua}_{\mathcal{A}}(x, y, p)$.

2. The **function extensionality** axiom implies that the typoid structure on $A \to B$ is univalent:

if $H, H' : f \simeq_{A\to B} g$ such that $H \cong_{A\to B} H'$, then $\texttt{funext}(H) = \texttt{funext}(H')$, since there is $p : H = H'$, hence $\texttt{ap}_{\texttt{funext}}(p) : \texttt{funext}(H) = \texttt{funext}(H')$.

3. By **UA** the typoid Uni is univalent.

If $(f, u), (g, w) : A \simeq_{\mathcal{U}} B$ such that $(f, u) \cong_{\mathcal{U}} (g, w)$, then $\texttt{ua}((f, u)) = \texttt{ua}((g, w))$, since

$$((f, u) =_{A\simeq_{\mathcal{U}} B} (g, w)) \simeq_{\mathcal{U}} \sum_{p:f=g} \left(p_*^{f\mapsto\texttt{isequiv}(f)}(u) = w \right).$$

By ext. $(f, u) \cong_{\mathcal{U}} (g, w)$ implies $f = g$, while a term of type $p_*^{f\mapsto\texttt{isequiv}(f)}(u) = w$ is found by equality of terms in $\texttt{isequiv}(g)$. $(f, u) \cong_{\mathcal{U}} (g, w)$ implies $(f, u) =_{A\simeq_U B} (g, w)$ and by application of $\texttt{ua}$ to get a term in $\texttt{ua}((f, u)) = \texttt{ua}((g, w))$.

Proposition

If $\mathcal{A}$ is a univalent typoid, the identity function $\mathrm{id}_A : A \to A$ is a typoid function from $\mathcal{A}$ to $\mathcal{A}_0$, with $\mathtt{Ua}^2_{\mathcal{A}}$ as a 2-associate of id_A w.r.t. its 1-associate $\mathtt{Ua}_{\mathcal{A}}$.

Theorem

Let $\mathcal{A}, \mathcal{B}$ be typoids and $f : A \to B$.

(i) If $\mathcal{A}$ is univalent, then f is a typoid function.
(ii) If $\mathcal{A}$ is strictly univalent, then f is a strict typoid function w.r.t. its 1-associate given in the proof of (i).

Proof.

$$x \simeq_{\mathcal{A}} y \xrightarrow{\mathtt{Ua}_{\mathcal{A}}(x,y)} x =_A y \xrightarrow{\mathtt{ap}_f(x,y)} f(x) =_B f(y)$$

$$\xrightarrow{\mathtt{IdtoEqv}_{\mathcal{B}}(f(x),f(y))} f(x) \simeq_{\mathcal{B}} f(y)$$

$$\Phi_f(x, y, e) \equiv \mathtt{IdtoEqv}_{\mathcal{B}}\Big(f(x), f(y), \mathtt{ap}_f(x, y, \mathtt{Ua}_{\mathcal{A}}(x, y, e))\Big).$$

$\square$

Theorem

If $\mathcal{A}, \mathcal{B}$ are univalent typoids, then $\mathcal{A} \times \mathcal{B}$ is a univalent typoid.

Proposition

If $\mathcal{A}, \mathcal{B}$ are typoids and $\mathcal{A} \times \mathcal{B}$ is univalent, then $\mathcal{A}, \mathcal{B}$ are univalent.

Definition

If $A : \mathcal{U}$, we call the typoid

$$\mathcal{A}^t \equiv (A, \simeq_{\mathcal{A}^t}, \mathrm{eqv}_{\mathcal{A}^t}, *_{\mathcal{A}^t}, {}^{-1}{}_{\mathcal{A}^t}, \cong_{\mathcal{A}^t})$$

truncated, if for every $x, y, z : A$, $e, e' : x \simeq_{\mathcal{A}^t} y$, and $d : y \sim_{\mathcal{A}^t} z$

$$x \simeq_{\mathcal{A}^t} y \equiv \mathbf{1},$$

$$\mathrm{eqv}_{\mathcal{A}^t}(x) \equiv 0_{\mathbf{1}},$$

$$*_{\mathcal{A}^t}(x, y, z, e, d) \equiv 0_{\mathbf{1}},$$

$${}^{-1}{}_{\mathcal{A}^t}(x, y, e) \equiv 0_{\mathbf{1}},$$

$$\cong_{\mathcal{A}^t}(x, y, e, e') \equiv (e = e').$$

The proof that $\mathcal{A}^t$ is a typoid is immediate. One needs only to take into account that $\mathtt{isProp}(\mathbf{1})$, hence $\mathtt{isSet}(\mathbf{1})$, where

$$\mathtt{isSet}(A) \equiv \prod_{x,y:A} \prod_{p,q:x=_A y} (p = q).$$

Proposition

If $A : \mathcal{U}$, $\mathcal{B}$ *is a typoid and* $f : B \to A$, *then* f *is a typoid function from* $\mathcal{B}$ *to* $\mathcal{A}^t$.

Corollary

If $A, B : \mathcal{U}$ *and* $f : B \to A$, *then* f *is a typoid function from* $\mathcal{B}^t$ *to* $\mathcal{A}^t$.

Proposition

If $A : \mathcal{U}$ *such that* $\texttt{isProp}(A)$, *then* $\mathcal{A}^t$ *is univalent.*

Corollary

If $A : \mathcal{U}$ *such that* $\texttt{isProp}(A)$, $\mathcal{B}$ *is a typoid and* $f : A \to B$, *then* f *is a typoid function from* $\mathcal{A}^t$ *to* $\mathcal{B}$.

Our second motivation for the study of typoids

Proposition

If $A : \mathcal{U}$, $\mathcal{B}$ *is a typoid such that* $\texttt{isProp}(B)$, *and* $f : A \to B$, *then* f *is a typoid function from* $\mathcal{A}^t$ *to* $\mathcal{B}$.

Proof.

By Corollary f is a typoid function from $\mathcal{A}^t$ to $\mathcal{B}^t$, while by Corollary id_B is a typoid function from $\mathcal{B}^t$ to $\mathcal{B}$. By composition of typoid functions $f \equiv \mathrm{id}_B \circ f$ is a typoid function from $\mathcal{A}^t$ to $\mathcal{B}$. □

In the setting of typoids we can **interpret the notion of the propositional truncation** $||A||$ **of a type** A **as the truncated typoid** $\mathcal{A}^t$.

Typoid-treatment for the HIT suspension ΣA of A.

If (A, a_0) is a pointed type, the **suspension typoid** of A is

$$\Sigma A = (\mathbf{2}, \simeq_{\Sigma A}, \mathrm{eq}_{\Sigma A}, *_{\Sigma A}, {}^{-1_{\Sigma A}}, \cong_{\Sigma A})$$

$$0 \simeq_{\Sigma A} 1 \equiv \sum_{f:2\to A} f(0) =_A a_0$$

$$1 \simeq_{\Sigma A} 0 \equiv \sum_{g:2\to A} g(1) =_A a_0$$

$$0 \simeq_{\Sigma A} 0 \equiv \mathbf{1} \equiv 1 \simeq_{\Sigma A} 1$$

$$\mathtt{merid} : A \to 0 \simeq_{\Sigma A} 1$$

$$\mathtt{merid}(x) \equiv (f_x, \mathtt{refl}_{a_0})$$

$$f_x(0) \equiv a_0, \quad f_x(1) \equiv x$$

Proposition

Let $\mathcal{B}$ be a typoid, $b_0, b_1 : B$, $m : A \to b_0 \simeq_{\mathcal{B}} b_1$, and let $f : \mathbf{2} \to B$ such that $f(0) \equiv b_0$ and $f(1) \equiv b_1$. Then f is a typoid function from ΣA to $\mathcal{B}$ with an 1-associate Φ_f satisfying

$$\Phi_f(0, 1, \mathtt{merid}(x)) \equiv m(x),$$

for every $x : A$.

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$$\mathtt{Typfun}(f) \equiv \sum_{\Phi_f : \prod_{x,y:A} \prod_{e:x \simeq_{\mathcal{A}} y} f(x) \simeq_{\mathcal{B}} f(y)} \Bigg[\Bigg(\prod_{x,y:A} \prod_{e:x \simeq_{\mathcal{A}} y} \prod_{d:y \simeq_{\mathcal{A}} z}$$

$$\Big(\Phi_f(x, x, \mathrm{eqv}_x) \cong_{\mathcal{B}} \mathrm{eqv}_{f(x)} \Big) \times$$

$$\Big(\Phi_f(x, z, e *_{\mathcal{A}} d) \cong_{\mathcal{B}} \Phi_f(x, y, e) *_{\mathcal{B}} \Phi_f(y, z, d) \Big) \Bigg) \times$$

$$\times \Bigg(\prod_{x,y:A} \prod_{e,d:x \simeq_{\mathcal{A}} y} \prod_{i:e \cong_{\mathcal{A}} d} \Phi_f(x, y, e) \cong_{\mathcal{B}} \Phi_f(x, y, d) \Bigg) \Bigg].$$

A canonical element of $\mathtt{Typfun}(f)$ is a pair $(\Phi_f, (U, \Phi_f^2))$, or for simplicity a triplet

$$(\Phi_f, U, \Phi_f^2),$$

where U is a term of the first type of the outer product and Φ_f^2 is a term of the second.

$$B^A \equiv \sum_{f:A\to B} \texttt{Typfun}(f).$$

If $\phi \equiv (f, \Phi_f, U, \Phi_f^2)$ and $\theta \equiv (g, \Phi_g, W, \Phi_g^2)$ are two canonical elements of B^A, we define

$$\phi \simeq_{B^A} \theta \equiv \sum_{\Theta_{f,g}:\prod_{x:A} f(x)\simeq_{\mathcal{B}} g(x)} \left(\prod_{x,y:A} \prod_{e:x\simeq_{\mathcal{A}} y} \Phi_{f(x,y,e)} *_{\mathcal{B}} \Theta_{f,g}(y) \cong_{\mathcal{B}} \right.$$

$$\left. \Theta_{f,g}(x) *_{\mathcal{B}} \Phi_g(x, y, e) \right).$$

A canonical element e of $\phi \simeq_{B^A} \theta$ is a pair $(\Theta_{f,g}, \Theta_{f,g}^2)$, where

$$\Theta_{f,g}^2 : \prod_{x,y:A} \prod_{e:x\simeq_{\mathcal{A}} y} \Phi_{f(x,y,e)} *_{\mathcal{B}} \Theta_{f,g}(y) \cong_{\mathcal{B}} \Theta_{f,g}(x) *_{\mathcal{B}} \Phi_g(x, y, e)$$

If ϕ is a canonical element of B^A we define $\mathrm{eqv}_\phi : \phi \simeq_{B^A} \phi$ as the pair $(\Theta_{f,f}, \Theta^2_{f,f})$, where

$$\Theta_{f,f} \equiv \lambda(x : A).\mathrm{eqv}_{f(x)} : \prod_{x:A} f(x) \simeq_{\mathcal{B}} f(x)$$

and $\Theta^2_{f,f}(x, y, e)$ proves the commutativity of the obvious diagram.

If $\phi \equiv (f, \Phi_f, U, \Phi^2_f), \theta \equiv (g, \Phi_g, W, \Phi^2_g), \eta \equiv (h, \Phi_h, V, \Phi^2_h)$ are canonical elements of B^A and $e \equiv (\Theta_{f,g}, \Theta^2_{f,g}) : \phi \simeq_{B^A} \theta$ and $d \equiv (\Theta_{g,h}, \Theta^2_{g,h}) : \theta \simeq_{B^A} \eta$, we define

$$e *_{B^A} d \equiv (\Theta_{f,h}, \Theta^2_{f,h}) : \phi \simeq_{B^A} \eta$$

$$\Theta_{f,h} \equiv \lambda(x : A).\Theta_{f,g}(x) *_{\mathcal{B}} \Theta_{g,h}(x),$$

and we can find $\Theta^2_{f,h}(x, y, e)$ of type

$$\Phi_{f(x,y,e)} *_{\mathcal{B}} \Theta_{f,h}(y) \cong_{\mathcal{B}} \Theta_{f,h}(x) *_{\mathcal{B}} \Phi_h(x, y, e).$$

If $e \equiv (\Theta_{f,g}, \Theta^2_{f,g}) : \phi \simeq_{B^A} \theta$, we define

$$e^{-1_{B^A}} \equiv (\Theta^{-1}_{f,g}, [\Theta^2_{f,g}]^{-1}) : \theta \simeq_{B^A} \phi,$$

where $\Theta^{-1}_{f,g} : \prod_{x:A} g(x) \simeq_{\mathcal{B}} f(x)$ is defined by

$$\Theta^{-1}_{f,g}(x) \equiv [\Theta_{f,g}(x)]^{-1_{\mathcal{B}}},$$

for every $x : A$, and $[\Theta^2_{f,g}]^{-1}(y, x, e)$ is a term of type

$$\Phi_g(y, x, e) *_{\mathcal{B}} \Theta_{f,g}(x)^{-1} \cong_{\mathcal{B}} \Theta_{f,g}(y)^{-1} *_{\mathcal{B}} \Phi_f(y, x, e).$$

Proposition

If $\mathcal{A}, \mathcal{B}$ are typoids, then $\mathcal{B}^{\mathcal{A}} = (B^A, \simeq_{B^A}, \mathrm{eqv}_{B^A}, *_{B^A}, ^{-1_{B^A}}, \cong_{B^A})$ is a typoid.

Proposition

If $\mathcal{A}, \mathcal{B}$ are typoids, then $\mathrm{ev}_{A,B} : B^A \times A \to B$, where $\mathrm{ev}_{A,B}((f, \Phi_f, U, \Phi^2_f), x) \equiv f(x)$ is a typoid function.

Expected: If $\mathcal{B}$ is univalent, then $\mathcal{B}^{\mathcal{A}}$ is univalent, and $\simeq$-form of CCC.