On relating type theories to (intuitionistic) set theories

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Scientific American, Quanta Magazine, Nautilus, ...

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Ian Hacking, *Why is there Philosophy of Mathematics at All?*, 2014.

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What are the set-theoretic counterparts (intuitionistic set theories) of such type theories?

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- Computational content: Witness and program extraction from proofs.
- Intuitionistically proved theorems hold in more generality: The internal logic of most topoi is intuitionistic logic.
- Axiomatic Freedom Adopt axioms that are classically refutable but interesting and desirable.

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- if all homotopically equivalent sets could be viewed as identical (univalence)?

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The rules are divided into formation, introduction, elimination and equality rules.

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- Empty represents falsum.
- Id(A, a, b) to represent equality on A.

The full system $\ensuremath{\mathsf{MLTT}}$

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- Denote by MLTT⁻ the theory MLTT without W-types.
- ► MLTT_n is the subsystem with only n universes U₀,...,U_{n-1}. Furthermore, MLTT_n⁻ also lacks the W-type constructor.

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Shall write $s =_A t$ rather than Id(A, s, t)

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In extensional type theory (Martin-Löf 1979, 1984) this hierarchy collapses, since $a =_A a'$ contains at most 1 element.

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In extensional type theory (Martin-Löf 1979, 1984) this hierarchy collapses, since $a =_A a'$ contains at most 1 element. Not so in intensional type theory (Martin-Löf 1973, 1986). Groupoid model (Hofmann, Streicher 1994), Kan simplicial sets (Voevodsky 2010), Kan cubical sets (Bezem, Coquand, Huber 2013).

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(Id–Formation)	$\Gamma \vdash A$ type	$\Gamma \vdash a : A$	$\Gamma \vdash b : A$
(Id-Formation)	$\Gamma \vdash a =_A b$ type		
(Id–Introduction)	$\frac{\Gamma \vdash a : A}{\Gamma \vdash 1_a : a = 1}$		
(Id-Uniqueness)	$\frac{\Gamma \vdash p : a}{\Gamma \vdash p = 1_a :}$		
(Id–Reflection)	$\frac{\Gamma \vdash p : a =_A}{\Gamma \vdash a = b :}$	$\frac{b}{A}$.	

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(Id–Reflection)	$\frac{\Gamma \vdash p : a =_A}{\Gamma \vdash a = b :}$	$\frac{b}{A}$.		

► Reflection makes judgemental identity undecidable, i.e., the (type checking) questions whether Γ ⊢ a = b : A or Γ ⊢ a : A hold become undecidable.

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i.e., if a, b : A and $p : a =_A b$ then $\widetilde{J}_d(a, b, p) : C(a, b, p)$ $d(a) = \widetilde{J}_d(a, a, 1_a) : C(a, a, 1_a)$

Rules for intensional identity

$$(\mathsf{Id}-\mathsf{Elim}) \qquad \begin{array}{l} \Gamma \vdash a : A \\ \Gamma \vdash b : A \\ \Gamma \vdash p : a =_A b \\ \Gamma, x : A, y : A, z : x =_A y \vdash C(x, y, z) \text{ type} \\ \overline{\Gamma, x : A \vdash d(x) : C(x, x, 1_x)} \\ \hline \Gamma \vdash \mathsf{J}(d, a, b, p) : C(a, b, p) \end{array}$$

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- Set Induction Scheme

Moving between type theory and set theory

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Aczel (late 1970's): The sets-as-trees interpretation (SaT)

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 R., S. Tupailo, Characterizing the interpretation of set theory in Martin-Löf type theory, APAL 2006.

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- ► Cesare Galozzi, Variations: Uses *h*-sets as index sets for the interpretation.

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Set Induction scheme

A set A is said to be regular if it is inhabited and transitive and whenever B ∈ A and R is a set relation such that ∀x ∈ B∃y ∈ A R(x, y) then there exists C ∈ A such that ∀x ∈ B∃y ∈ C R(x, y) and ∀y ∈ C∃x ∈ B R(x, y).

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- A set *I* is said to be weakly inaccessible if *I* is a regular set such that *I* |= CZF⁻.
- A set *I* will be called **inaccessible** if *I* is weakly inaccessible and for all *x* ∈ *I* there exists a regular set *y* ∈ *I* such that *x* ∈ *y*.

An 'algebraic' characterization of "inaccessibility"

Proposition (CZF $^-$)

A set I is weakly inaccessible iff I is a regular set such that the following are satisfied:

- 1. $\omega \in I$,
- 2. $\forall a \in I \cup a \in I$,
- 3. $\forall a \in I \ [a \text{ inhabited } \Rightarrow \bigcap a \in I]$,
- 4. $\forall A, B \in I \exists C \in I \quad C \text{ is full in } \mathbf{mv}(^{A}B).$

Recall that \mathbf{CZF}^- denotes the theory \mathbf{CZF} without the Set Induction scheme.

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Theorem 1. (Crosilla, R. 2002)

The theory

 $CZF^- + \forall x \exists I [x \in I \land I \text{ weakly inaccessible}]$

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Proposition. MLTT⁻ can be interpreted in

CZF + weak-INACC

where weak-INACC stands for $\forall x \exists I [x \in I \land I \text{ weakly inaccessible}].$

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Corollary. All the theories $MLTT^-$, CZF + weak-INACC, and $MLTT^-$ + UA are of the same strength.

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Corollary. All the theories $MLTT^-$, CZF + weak-INACC, and $MLTT^-$ + UA are of the same strength.

It does not matter whether the identity type is extensional or intensional.

It was known by work of Jervell 1978 and Feferman 1980 that (extensional) **MLTT**⁻ has strength Γ_0 .

Let f, g : ∏_{x:A} P(x). A homotopy from f to g is a dependent function of type

$$(f \simeq g) :\equiv \prod_{x:A} (f(x) =_{P(x)} g(x)).$$

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$$\mathsf{isequiv}(f) :\equiv (\sum_{g:B \to A} (f \circ g \simeq \mathsf{id}_B)) \times (\sum_{h:B \to A} (h \circ f \simeq \mathsf{id}_A)).$$

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$$(A \simeq B) := \sum_{f:A \to B} isequiv(f).$$

For types A, B : U there is a canonical function

$$idtoeqv: (A =_{\mathcal{U}} B) \vdash (A \simeq B).$$

The **Univalence Axiom** asserts that this function is itself an equivalence:

$$(A =_{\mathcal{U}} B) \simeq (A \simeq B).$$

Theorem:

The following theories prove the same arithmetical statements:

(i) MLTT.

(ii) The extensional type theory **MLTT**^{ext}.

(iii) **CZF** plus for every $n \in \mathbb{N}$, an axiom asserting that there is a tower of n-many inaccessible sets, **CZF** + $\bigcup_n INACC_n$.

(iv) **CZF** + $\bigcup_n INACC_n + RDC + Presentation Ax,$

where RDC signifies the relativized dependent choices axiom.

It's the same as

 $\mathsf{KP} + \{n\text{-many recursively inaccessible ordinals}\}_{n \in \mathbb{N}}$

or

 Δ_2^1 -CA + {*n* tower of β -models of Δ_2^1 -CA} $_{n \in \mathbb{N}}$

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- ► The strength of all of these theories is considerable but tiny when compared to Π¹₂-CA₀.
- Does the addition of the Univalence Axiom change that picture?
- No, since the cubical model of Bezem, Coquand, Huber can be done "constructively" in type theory, though not all types have been included yet.

For details see M. Rathjen *Proof Theory of Constructive Systems: Inductive Types and Univalence*, arXiv:1610.02191 (2016).

Vicious circles

"... vicious circles ... [arise] from supposing that a collection of objects may contain members which can only be defined by means of the collection as a whole. [....] We shall, therefore, have to say that statements about 'all propositions' are meaningless. By saying that a set has 'no total,' we mean, primarily, that no significant statement can be made about 'all its members.' In such cases, it is necessary to break up our set into smaller sets, each of which is capable of a total. This is what the theory of types aims at effecting." Whitehead & Russell So we must be very careful about introducing the notion of proposition.

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- There are predicative approaches to this which lead to level restrictions as in Principia and allow only "smaller collections" into which **Prop** is broken, such as Martin-Löf's **universes**.

- So we must be very careful about introducing the notion of proposition.
- There are predicative approaches to this which lead to level restrictions as in Principia and allow only "smaller collections" into which **Prop** is broken, such as Martin-Löf's **universes**.
- Or one sticks to the impredicative approach but restricts the type forming operations in other ways as for instance done in system F.

We shall, therefore, not assume anything of what may seem to be involved in the common-sense admission of classes, except this, that every propositional function is equivalent, for all its values, to some predicative function of the same arguments. [...] We will call this assumption the axiom of classes, or the axiom of reducibility.

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$$\begin{split} \varphi \land \psi &\equiv \forall p[(\varphi \to (\psi \to p)) \to p] \\ \varphi \lor \psi &\equiv \forall p[(\varphi \to p) \to ((\psi \to p) \to p)] \end{split}$$

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in terms of \rightarrow and \forall via quantification over propositions $\forall p$:

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 Prawitz showed in (1965) that the above equivalences hold in second order intuitionist logic.

In fact, the above equivalences can be used as definitions in the →, ∀ fragment of second order intuitionistic logic, thereby reducing full intuitionistic second order logic to this fragment.

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- This idea is also used to express logic in Girard's system F (1971) and is the standard approach to representing logic in the calculus of constructions (Coquand 1990) and extensions.
- The standard approach to representing logic in the type theory Lego (Luo & Pollack 1992; Luo 1994) and also, sometimes, the type theory of Coq (Barras et al. 1996), is to use the above Russell-Prawitz representation, where the variable p ranges over the the impredicative type called

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$$\frac{\Gamma, x : A \vdash B(x) : \mathbf{Prop}}{\Gamma \vdash \prod_{x : A} B(x) : \mathbf{Prop}}$$

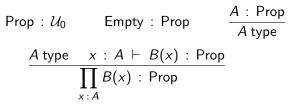
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Note that this rule is highly impredicative as A can be any type (e.g. Prop).

$\mathsf{Prop} : \mathcal{U}_0 \qquad \mathsf{Empty} : \mathsf{Prop} \qquad \frac{A : \mathsf{Prop}}{A \mathsf{type}}$

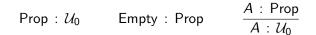


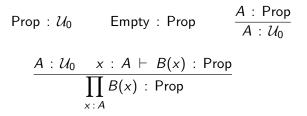
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- Let's treat restricted cases first.





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Now let's stick to one universe but strengthen the rules for Prop so that it reflects all types A.

Prop embodies Powerset

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Now let's stick to one universe but strengthen the rules for Prop so that it reflects all types A.

$$\frac{A: \text{type } x: A \vdash B: \text{Prop}}{\prod_{x:A} B(x): \text{Prop}}$$

Extensionality

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 $(\mathit{IND}_{\in}) \quad \forall a \ (\forall x \in a \ \varphi(x) \ \rightarrow \ \varphi(a)) \ \rightarrow \ \forall a \ \varphi(a),$

► IZF has the same strength as ZF (Friedman).

Gambino 2000

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Negative Separation

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Negative Power Set

 $\exists z \,\forall x \, [x \in z \,\leftrightarrow\, x \subseteq a \,\land\, \forall u \in a \, (\neg \neg u \in x \vdash u \in x)]$

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 Then V(T) ⊨ CZF but refutes Powerset.
- $V(\mathcal{T}) \models$ negative Powerset + negative Separation.

The strength of $\textbf{IZF}^\neg\neg$

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Conjecture: IZF^{¬¬} is much weaker in strength than **ZFC**.

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Axiom of Propositional Resizing

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 $egin{array}{lll} \Omega := {\sf Prop}_{{\mathcal U}_0} \ {\mathcal P}({\mathcal A}) \ := \ ({\mathcal A} o \Omega). \end{array}$

Grazie mille