On relating type theories to (intuitionistic) set theories

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- What are the set-theoretic counterparts (intuitionistic set theories) of such type theories?

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- Axiomatic Freedom Adopt axioms that are classically refutable but interesting and desirable.

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- if all homotopically equivalent sets could be viewed as identical (univalence)?

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The rules are divided into formation, introduction, elimination and equality rules.

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- $\mathrm{Id}(A, a, b)$ to represent equality on A.

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- A universes is a type inhabited by types. Every universe is closed under all the previous type constructions and $\mathcal{U}_i : \mathcal{U}_{i+1}$.

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- Denote by $\mathbf{MLTT}^{-}$ the theory $\mathbf{MLTT}$ without W-types.
- $\mathbf{MLTT}_n$ is the subsystem with only n universes $\mathcal{U}_0, \ldots, \mathcal{U}_{n-1}$.

  Furthermore, $\mathbf{MLTT}_n^{-}$ also lacks the W-type constructor.

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  Shall write $s =_A t$ rather than $\mathsf{Id}(A, s, t)$

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Not so in intensional type theory (Martin-Löf 1973, 1986). Groupoid model (Hofmann, Streicher 1994), Kan simplicial sets (Voevodsky 2010), Kan cubical sets (Bezem, Coquand, Huber 2013).

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(Id–Formation) $$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash a =_A b \text{ type}}$$

(Id–Introduction) $$\frac{\Gamma \vdash a : A}{\Gamma \vdash 1_a : a =_A a}$$

(Id–Uniqueness) $$\frac{\Gamma \vdash p : a =_A b}{\Gamma \vdash p = 1_a : a =_A b}$$

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$$\text{(Id–Reflection)} \qquad \frac{\Gamma \vdash p : a =_A b}{\Gamma \vdash a = b : A}.$$

- Reflection makes judgemental identity undecidable, i.e., the (type checking) questions whether $\Gamma \vdash a = b : A$ or $\Gamma \vdash a : A$ hold become undecidable.

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i.e., if $a, b : A$ and $p : a =_A b$ then

$$\begin{aligned} \tilde{\mathrm{J}}_d(a, b, p) &\ :\ C(a, b, p) \\ d(a) = \tilde{\mathrm{J}}_d(a, a, 1_a) &\ :\ C(a, a, 1_a) \end{aligned}$$

Rules for intensional identity

$$
(\text{Id–Elim}) \quad \frac{\begin{array}{l} \Gamma \vdash a : A \\ \Gamma \vdash b : A \\ \Gamma \vdash p : a =_A b \\ \Gamma, x : A, y : A, z : x =_A y \vdash C(x, y, z) \text{ type} \\ \Gamma, x : A \vdash d(x) : C(x, x, 1_x) \end{array}}{\Gamma \vdash \mathsf{J}(d, a, b, p) : C(a, b, p)}
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(\text{Id–Eq}) \quad \frac{\begin{array}{l} \Gamma \vdash a : A \\ \Gamma, x : A, y : A, p : x =_A y \vdash C(x, y, p) \text{ type} \\ \Gamma, x : A \vdash d(x) : C(x, x, 1_x) \end{array}}{\Gamma \vdash \mathsf{J}(d, a, a, 1_a) = d(a) : C(a, a, 1_a)}
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- **Replacement**

Myhill's Constructive set theory 1975

CST based on intuitionistic logic

Many sorted system: **numbers, sets, functions**

Axioms (simplified)

- **Extensionality**
- **Pairing**, **Union**, **Infinity** (or $\mathbb{N}$ is a set)
- **Bounded Separation**
- **Exponentiation**: A, B sets $\Rightarrow A^B$ set.
- **Replacement**
- **Set Induction Scheme**

Moving between type theory and set theory

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The types-as-sets interpretation (TaS).

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Aczel (late 1970's): The sets-as-trees interpretation (SaT)

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- R., S. Tupailo, Characterizing the interpretation of set theory in Martin-Löf type theory, APAL 2006.

- R., S. Tupailo, Characterizing the interpretation of set theory in Martin-Löf type theory, APAL 2006.
- Cesare Galozzi, Variations: Uses h-sets as index sets for the interpretation.

Constructive Zermelo-Fraenkel set theory, **CZF**

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- **Strong Collection**

$$
\begin{aligned}
&(\forall x \in a)\, \exists y \; \varphi(x, y) \; \rightarrow \\
&\exists b \, [\, (\forall x \in a)\, (\exists y \in b) \; \varphi(x, y) \; \wedge \; (\forall y \in b)\, (\exists x \in a) \; \varphi(x, y) \,]
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Three notions of large set

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- A set A is said to be **regular** if it is inhabited and transitive and whenever $B \in A$ and R is a set relation such that $\forall x \in B\, \exists y \in A\ R(x, y)$ then there exists $C \in A$ such that $\forall x \in B\, \exists y \in C\ R(x, y)$ and $\forall y \in C\, \exists x \in B\ R(x, y)$.

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- A set I is said to be **weakly inaccessible** if I is a regular set such that $I \models \mathbf{CZF}^-$.
- A set I will be called **inaccessible** if I is weakly inaccessible and for all $x \in I$ there exists a regular set $y \in I$ such that $x \in y$.

An 'algebraic' characterization of "inaccessibility"

Proposition ($\mathbf{CZF}^-$)

A set I is weakly inaccessible iff I is a regular set such that the following are satisfied:

1. $\omega \in I$,
2. $\forall a \in I \; \bigcup a \in I$,
3. $\forall a \in I \, [a \text{ inhabited } \Rightarrow \bigcap a \in I]$,
4. $\forall A, B \in I \, \exists C \in I \quad C$ is full in $\mathbf{mv}({}^{A}B)$.

How strong is $\mathbf{MLTT}^{-}$ plus Univalence?

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Recall that **CZF**$^-$ denotes the theory **CZF** without the Set Induction scheme.

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Theorem 1. (Crosilla, R. 2002)

The theory

$$\mathbf{CZF}^- + \forall x\, \exists I\, [x \in I \ \wedge \ I \text{ weakly inaccessible}]$$

has the same strength as

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so has proof-theoretic ordinal Γ_0.

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Proposition. $\mathbf{MLTT}^-$ can be interpreted in

$$\mathbf{CZF} + \text{weak-INACC}$$

where weak-INACC stands for
$\forall x\, \exists I\, [x \in I \ \wedge\ I \text{ weakly inaccessible}]$.

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Corollary. All the theories $\mathbf{MLTT}^-$, $\mathbf{CZF} + \text{weak-INACC}$, and $\mathbf{MLTT}^- + \mathbf{UA}$ are of the same strength.

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It does not matter whether the identity type is extensional or intensional.

It was known by work of Jervell 1978 and Feferman 1980 that (extensional) $\mathbf{MLTT}^-$ has strength Γ_0.

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- $(A \simeq B) \;:\equiv\; \sum_{f:A\to B} \mathsf{isequiv}(f)$.
- For types $A, B : \mathcal{U}$ there is a canonical function

$$\mathsf{idtoeqv} : (A =_{\mathcal{U}} B) \vdash (A \simeq B).$$

  The **Univalence Axiom** asserts that this function is itself an equivalence:

$$(A =_{\mathcal{U}} B) \simeq (A \simeq B).$$

Strength of **MLTT**

Theorem:

The following theories prove the same arithmetical statements:

(i) **MLTT**.

(ii) *The extensional type theory* $\mathbf{MLTT}^{ext}$.

(iii) **CZF** *plus for every* $n \in \mathbb{N}$, *an axiom asserting that there is a tower of n-many inaccessible sets,* $\mathbf{CZF} + \bigcup_n \mathrm{INACC}_n$.

(iv) $\mathbf{CZF} + \bigcup_n \mathrm{INACC}_n + \mathrm{RDC} +$ *Presentation Ax,*

where RDC *signifies the* relativized dependent choices axiom.

"Classical" Strength of **MLTT**

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- It's the same as

$$\mathbf{KP} + \{n\text{-many recursively inaccessible ordinals}\}_{n \in \mathbb{N}}$$

or

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- The strength of all of these theories is considerable but tiny when compared to Π^1_2-$\mathbf{CA}_0$.
- Does the addition of the Univalence Axiom change that picture?
- No, since the cubical model of **Bezem, Coquand, Huber** can be done "constructively" in type theory, though not all types have been included yet.

  For details see M. Rathjen *Proof Theory of Constructive Systems: Inductive Types and Univalence*, arXiv:1610.02191 (2016).

Vicious circles

"... vicious circles ... [arise] from supposing that a collection of objects may contain members which can only be defined by means of the collection as a whole. [....] We shall, therefore, have to say that statements about 'all propositions' are meaningless. By saying that a set has 'no total,' we mean, primarily, that no significant statement can be made about 'all its members.' In such cases, it is necessary to break up our set into smaller sets, each of which is capable of a total. This is what the theory of types aims at effecting." Whitehead & Russell

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- There are predicative approaches to this which lead to level restrictions as in Principia and allow only “smaller collections” into which **Prop** is broken, such as Martin-Löf’s **universes**.

- So we must be very careful about introducing the notion of proposition.
- There are predicative approaches to this which lead to level restrictions as in Principia and allow only “smaller collections” into which **Prop** is broken, such as Martin-Löf’s **universes**.
- Or one sticks to the impredicative approach but restricts the type forming operations in other ways as for instance done in system **F**.

We shall, therefore, not assume anything of what may seem to be involved in the common-sense admission of classes, except this, that every propositional function is equivalent, for all its values, to some predicative function of the same arguments. [...] We will call this assumption the axiom of classes, or the axiom of reducibility.

The Russell-Prawitz interpretation of logic

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\end{aligned}$$
- Prawitz showed in (1965) that the above equivalences hold in second order intuitionist logic.

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- This idea is also used to express logic in Girard's system **F** (1971) and is the standard approach to representing logic in the **calculus of constructions** (Coquand 1990) and extensions.
- The standard approach to representing logic in the type theory **Lego** (Luo & Pollack 1992; Luo 1994) and also, sometimes, the type theory of **Coq** (Barras et al. 1996), is to use the above **Russell-Prawitz** representation, where the variable p ranges over the the impredicative type called

  Prop

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- Propositions are represented as objects of type **Prop**. These objects are themselves types (or names of types in the Tarski treatment).

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- Note that this rule is highly impredicative as A can be any type (e.g. **Prop**).

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$$\frac{A : \mathsf{Prop} \quad b_1 : A \quad b_2 : A}{b_1 = b_2 : A}$$

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- Let's treat restricted cases first.

The type Prop reflecting just $\mathcal{U}_0$

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Now let's stick to one universe but strengthen the rules for Prop so that it reflects all types A.

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$$\frac{A : \mathsf{type} \quad x : A \vdash B : \mathsf{Prop}}{\prod_{x:A} B(x) : \mathsf{Prop}}$$

Intuitionistic Zermelo-Fraenkel set theory, **IZF**

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- **Collection**

$$(\forall x \in a)\, \exists y\ \varphi(x, y) \quad \vdash \quad \exists b\ (\forall x \in a)\, (\exists y \in b)\ \varphi(x, y)$$

Intuitionistic Zermelo-Fraenkel set theory, **IZF**

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- **Set Induction**

$$(IND_{\in}) \qquad \forall a\ (\forall x \in a\ \varphi(x) \ \rightarrow\ \varphi(a)) \ \rightarrow\ \forall a\ \varphi(a),$$

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- **IZF** has the same strength as **ZF** (Friedman).

Two set-theoretic axioms pertaining to **Prop**

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$$\exists y\, \forall x\, [x \in y \;\leftrightarrow\; x \in a \wedge \neg\neg\varphi(x)]$$

for all formulae $\varphi(x)$.

Two set-theoretic axioms pertaining to **Prop**

- Gambino 2000
- **Negative Separation**

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  for all formulae $\varphi(x)$.
- **Negative Power Set**

$$\exists z \, \forall x \, [x \in z \; \leftrightarrow \; x \subseteq a \, \wedge \, \forall u \in a \, (\neg\neg u \in x \vdash u \in x)]$$

Negative Intuitionistic Zermelo-Fraenkel set theory, $\mathbf{IZF}^{\neg\neg}$

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Conjecture: $\mathbf{IZF}^{\neg\neg}$ is much weaker in strength than **ZFC**.

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Grazie mille