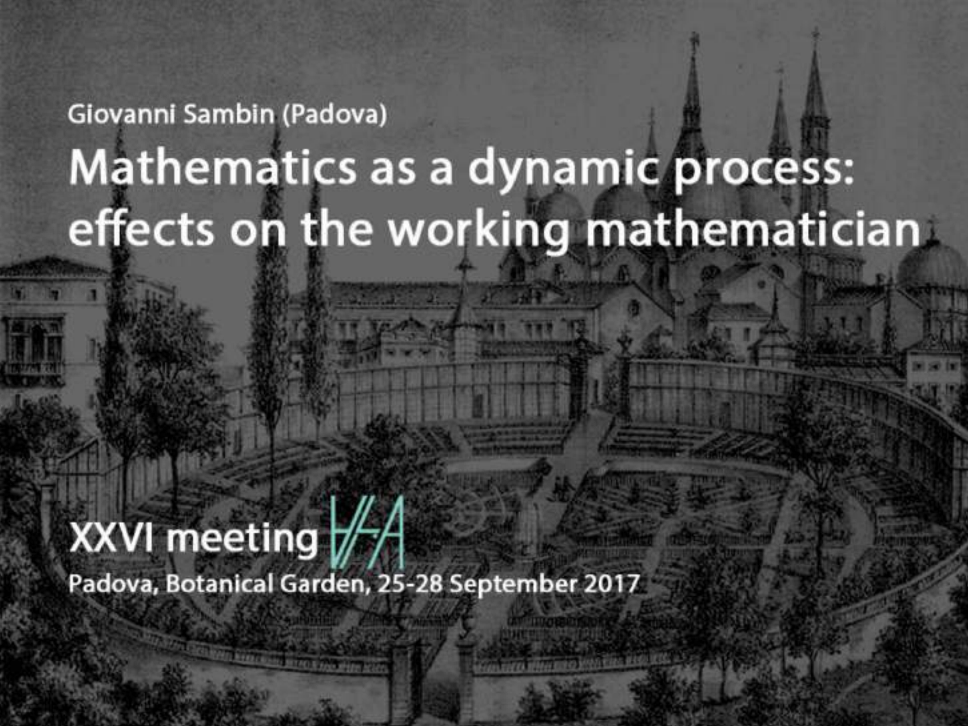


Giovanni Sambin (Padova)

Mathematics as a dynamic process: effects on the working mathematician

XXVI meeting

Padova, Botanical Garden, 25-28 September 2017

The problem of foundations - summary

- non-euclidean geometry: loss of absolute truth in geometry
- abstract algebra (to cope with complexity)
- rigorization of analysis ("pathological curves",...)

- Cantor, Dedekind, Frege, Peano: naive set theory
- Paradoxes, i.e. **contradictions**: Burali-Forti 1896, Russel 1901, ...
- Crisis of foundations
- Traditional ways out:
  - logicism (Frege, Russell, Whitehead, Principia Mathematica 1911)
  - constructivism (Kronecker, Borel, Poincaré, Brouwer, Heyting,...)
  - formalism (Hilbert, Zermelo,...)

Hilbert's program: consistency of ZFC, a finitary proof

We don't have a proof of formal consistency of ZFC,
and most probably we will never have one:

$ZFC \nvdash Con(ZFC)$ by Gödel's 2nd incompleteness theorem

Common paradigm today

Somehow paradoxically... the common paradigm is:

Bourbaki's attitude = denial of the problem

platonist on weekdays, formalist on sundays split mind

... when philosophers attack... we rush to hide behind formalism and say "mathematics is just a combination of meaningless symbols"... we are left in peace... with the feeling each mathematician has that he is working with something real. This sensation is probably an illusion, but is very convenient. That is Bourbaki's attitude toward foundations.

J. A. Dieudonné, 1970, see Davis-Hersh *The mathematical experience*, 1981

Formally classical logic and axiomatic set theory ZFC, ignoring Gödel.
An **act of faith** remains necessary

Authority and conservativity

One should not have faith in some kind of ideal **authority** from which something follows which is clearly wrong from a more concrete point of view.

conservativity of ideals over real life is missing:

- in religion: slavery, wars, female genital mutilation,...
- in economy: destruction of the planet
- in mathematics: Banach - Tarski, wellordering of real numbers

The reason why non-conservativity is common is that one looks for bivalence, completeness, aiming to achieve safety through **static stability**.

A change of paradigm

Open minded meditation on Göödel's incompleteness, as opposed to Bourbaki's dogmatic denial of its significance, leads naturally to conceive mathematics as

dynamic, partial, plural, that is, a conquered human achievement,

instead of

static, complete, unique, that is, a given absolute truth.

In other words, it becomes possible to see mathematics as produced by a Darwinian process of evolution, as all other fields of science.

By the same reason, truth and mathematics cannot be reduced to one system chosen once for all.

Truth and mathematics remain something to conquer, step by step.

Whenever we try to block it, by Göödel's results it becomes incomplete.

Martin-Löf constructive type theory

full control of information (intensional, computer formalization)

provably consistent (procedure for normalization of proofs)

Dynamic constructivism

Abandoning absolute truth, acts of faith in authority

looking for a perspective which allows conservativity

leads to a natural, dynamic view.

"It is now generally believed by biologists and by neuroscientists that nothing beyond biology and evolution is theoretically necessary to explain human beings, that is, their bodies and their minds."

"My general claim is simply that the same holds for logic and mathematics. That is, a naturalistic, evolutionary attitude is fully sufficient, and [...] actually convenient for explaining not only the human body and mind, but also all human intellectual products, including logic and mathematics, which are just the most exact of these. That is, mathematics is a product of our minds and so to explain it we require no more than what is needed by biologists to explain the mind."

GS, Steps toward a dynamic constructivism, 2003

With hindsight, this is what produces a paradigm shift, in the sense of

Th. Kuhn, The structure of scientific revolutions, 1962, 1970 it is **simpler** and **more effective** to manipulate symbols than things:

$$\begin{array}{ccccc} & & \text{mathematical manipulation} & & \\ & & \longrightarrow & & \\ \text{abstraction} & \uparrow & & \downarrow & \text{application} \\ & & \longrightarrow & & \\ & & \text{reality} & & \end{array}$$

Mathematics is the exploration of notions and structures of our abstract (reliable) thought which can be useful to understand the world.
Every culture has its own mathematics; it is useful to man for survival, a continuation of natural **evolution**.

some consequences:

- every notion is the result of human abstraction
- many ways to abstract = many kinds of mathematics, pluralism
- application is part of mathematics
- objectivity is a result, not a cause; dynamic, evolutionary view of mathematics

Adopting dynamic constructivism means adhering to the principles:

1. Cultivate pluralism in mathematics and foundations.
Different styles in abstraction, which means different foundations, produce different kinds of mathematics and should be respected.

2. Accept open notions and incomplete theories.
The construction of mathematics is a never-ending process and nothing is given in advance

no fixed universes of all sets, or of all subsets, or of all propositions.

3. Preserve all conceptual distinctions (no reductionism).

many more primitive notions than usual

4. Preserve all different levels of abstraction.

real/ideal mathematics

extensional/intensional foundational system

language/metalanguage

Novelties in mathematics

We get rid of the oppression of an absolute and complete truth, and of some transcendent authority

it's only our responsibility

We become more relaxed, free and creative

Since nothing is assumed to exist before us, we should be aware in every moment of the abstractions by which we pass from reality to mathematical notions.

price: one has to start again from the beginning: main task is to find correct constructive definitions

most interesting, fascinating **reward**: several novelties emerge which were **hidden** by stronger foundations, using `PSA`, `LEM`

surprise: the **extra information** which we must keep has a clear logical structure, it is not "code"

Total control of information is not good for the working mathematician.

e.g. extensional equality of functions

To define subsets and relations we need to forget some piece of information, following the

forget - restore principle

Each logical constant (connectives and quantifiers) is the reflection at the (more abstract) object language level of a link between assertions at the (more concrete) metalevel.

Principle of reflection

We will see another 2 examples of this (real-ideal, MF with two levels of abstraction)

duality and symmetry in topology

Management of information means we distinguish two cases:

Set: we have effective rules to obtain all elements

Collection: we don't

Hence we do not assume Power Set Axiom $\mathtt{PSA}$.

To produce a topology ΩX on a set X without $\mathtt{PSA}$ it is necessary to start from a base for open subsets $\mathrm{ext}\,(a) \subseteq X$ $(a \in S)$ indexed on a second set S.

Equivalently $(X, \Vdash, S)$ where $x \Vdash a \equiv x \,\epsilon\, \mathrm{ext}\,(a)$.

We use the relation overlap between subsets $D, E \subseteq X$:

$$E \between D \equiv (\exists x \in X)(x \,\epsilon\, E \;\&\; x \,\epsilon\, D)$$

Then interior and closure of $D \subseteq X$ are defined by:

$$x \,\epsilon\, \mathrm{int}\, D \equiv \exists a(x \Vdash a \;\&\; \mathrm{ext}\, a \subseteq D)$$

$$x \,\epsilon\, \mathrm{cl}\, D \equiv \forall a(x \Vdash a \rightarrow \mathrm{ext}\, a \between D)$$

Since $E \between D$ is the logical dual of $E \subseteq D \equiv (\forall x \in X)(x \,\epsilon\, E \rightarrow x \,\epsilon\, D)$, we find that

int and cl are defined by strictly dual formulas,

obtained one from another by swapping $\forall$, $\exists$ and $\rightarrow$, $\&$.

By looking at definitions, one can see that int and cl are obtained by composing more elementary operators between $\mathcal{P}X$ and $\mathcal{P}S$. Putting:

$$x \,\epsilon\, \mathsf{ext}\, U \equiv \Diamond x \between U, \quad a \,\epsilon\, \Diamond D \equiv \mathsf{ext}\, a \between D,$$

$$x \,\epsilon\, \mathsf{rest}\, U \equiv \Diamond x \subseteq U, \quad a \,\epsilon\, \Box D \equiv \mathsf{ext}\, a \subseteq D,$$

then

$$\mathsf{int} = \mathsf{ext}\,\Box, \quad \mathsf{cl} = \mathsf{rest}\,\Diamond$$

With no conditions on $\Vdash$, the structure $(X, \Vdash, S)$ is perfectly symmetric.

So we define the operators $\mathcal{J}, \mathcal{A}$ on $\mathcal{P}S$ symmetric of $\mathsf{int}\,, \mathsf{cl}$:

$$\mathcal{J} = \Diamond\, \mathsf{rest}\,, \quad \mathcal{A} = \Box\, \mathsf{rest}\ .$$

Since $\mathsf{ext} \dashv \Box$ and $\Diamond \dashv \mathsf{rest}$ are adjunctions:

$\mathsf{int}\,, \mathcal{J}$ are reductions (contractive, monotone, idempotent)

$\mathsf{cl}\,, \mathcal{A}$ are saturations (expansive, monotone, idempotent).

Moreover, open subsets of X, i.e. fixed points for int, coincide with those of the form $\mathsf{ext}\, U$ for some $U \subseteq S$.

They form a complete lattice which is isomorphic to fixed points for $\mathcal{A}$, which are hence called formal open subsets.

All this applies dually to closed subsets.

X set of points, $D \subseteq X$ — S set of observables, $U \subseteq S$

$\exists\forall$ *reduction* — *concrete open* $\operatorname{int} D = D$ — *formal closed* $U = \mathcal{J} U$

$$\begin{tikzcd}
\operatorname{ext} U \arrow[rr, dash, "\textit{symmetric}"] \arrow[dd, dash, "\textit{dual}"'] \arrow[ddrr, <->, "\textit{isomorphic}"] & & \Diamond D \arrow[dd, dash, "\textit{dual}"] \arrow[ddll, <->, "\textit{isomorphic}"'] \\
& & \\
\operatorname{rest} U \arrow[rr, dash, "\textit{symmetric}"'] & & \Box D
\end{tikzcd}$$

$\forall\exists$ *saturation* — $\operatorname{cl} D = D$ *concrete closed* — $U = \mathcal{A} U$ *formal open*

These discoveries were buried under excess of assumptions:

- `LEM` forces validity $\operatorname{cl} = -\operatorname{int} -$
- `PSA` makes the second set S useless (since ΩX itself is a set, and $\Vdash = \in$).

This is sufficient to reject the claim: classical paradigm = “absolute truth”


ALZANO

Convergence as the only mathematical module over fully logical (structural) definitions

Up to here the structure is a basic pair $(X, \Vdash, S)$ where $\Vdash$ is **any** relation.

Recall that the open subsets are of the form $\mathrm{ext}\, U \equiv \cup_{b \epsilon U} \mathrm{ext}\, b$.

A topological space is a set X of points and a collection of open subsets ΩX closed under arbitrary unions and finite intersections

$$(X, \Omega X) \quad \Rightarrow \quad (X, \epsilon, \Omega X) \quad \Rightarrow \quad (X, \Vdash, S)$$

where $\mathrm{ext}\, a \subseteq X (a \in S)$ is a base for ΩX and $x \Vdash a \equiv x \ \epsilon \ \mathrm{ext}\, a$.
Open subsets form a topology ΩX iff $\mathrm{ext}\,(a) \subseteq X \ (a \in S)$ is a base, that is, satisfies convergence:

B1: $\mathrm{ext}\, a \cap \mathrm{ext}\, b = \mathrm{ext}\,(a \downarrow b)$

where $c \ \epsilon \ a \downarrow b \equiv \mathrm{ext}\, c \subseteq \mathrm{ext}\, a \ \& \ \mathrm{ext}\, c \subseteq \mathrm{ext}\, b$

every two neighbourhoods have a common refinement

B2: $\mathrm{ext}\, S = X$

every point has a neighbourhood

Having bases S, T, we discover that a function $f : X \to Y$ is continuous from $(X, \Vdash, S)$ into $(Y, \Vdash', T)$ iff there is a relation $s : S \to T$ s.t. $\Vdash' \circ f = s \circ \Vdash$.
By symmetry, it is natural to consider a relation r also between X and Y.
Then t.f.a.e.:

1. r is continuous, that is $r\,x \between \mathrm{ext}\, b \to \exists a(x \Vdash a \;\&\; \mathrm{ext}\, a \subseteq r^{-}\, \mathrm{ext}\, b)$
2. r^{-} is open,
3. there exists a relation $s : S \to T$ such that $r^{-}\,\mathrm{ext}\, b = \mathrm{ext}\, s^{-} b$ for all $b \in T$.

Continuity becomes $\Vdash' \circ r = s \circ \Vdash$, a commutative square of relations :

$$\begin{tikzcd} X \arrow[r, "\Vdash"] \arrow[d, "r"'] & S \arrow[d, "s"] \\ Y \arrow[r, "\Vdash'"] & T \end{tikzcd}$$

categories **BP** and **CSpa**

BP: basic pairs $(X, \Vdash, S)$ and relation-pairs (r, s) (commutative squares)

CSpa: concrete spaces = convergent basic pairs, that is: B1-B2 hold, or equivalently every $\mathrm{cl}\,\{x\}$ is convergent

relation-pairs (r, s) preserving convergence:

r maps convergent subsets into convergent subsets

convergent subset = topological notion of point

r^- respects finite intersections

$r^- \,\mathrm{ext}\,(b \downarrow_{\mathcal{Y}} c) = \mathrm{ext}\,(s^- b \downarrow_{\mathcal{X}} s^- c)$, for all $b, c \in T$, and $r^- \,\mathrm{ext}\, T = \mathrm{ext}\, S$,

Pointfree topology

The pointfree approach to topology is a **must**.

In many cases, points do not form a set. So we must obtain them as ideal points over an effective, pointfree structure

(exactly as: real numbers are ideal points over unions of intervals with rational end-points)

general methodology: **axiomatise the structure induced on the formal side**

basic pair $\mathcal{X}$ $\Rightarrow$ basic topology $Fs(\mathcal{X})$

$(r, s) : \mathcal{X} \to \mathcal{Y}$ relation-pair $\Rightarrow$ s continuous relation from $Fs(\mathcal{X}) \to Fs(\mathcal{Y})$

in general, $s : \mathcal{S} \to \mathcal{T}$ if s preserves $\mathcal{J}$ and $s - *$ preserves $\mathcal{A}$

concrete space $\Rightarrow$ positive topology

$(r, s) : \mathcal{X} \to \mathcal{Y}$ convergent relation-pair

$\Rightarrow$ s formal map $Fs(\mathcal{X}) \to Fs(\mathcal{Y})$

singleton $\{x\}$, or equivalently convergent subset D

$\Rightarrow$ $\Diamond x$ and $\Diamond D$ ideal point of $Fs(\mathcal{X})$

Basic topology: axiomatization of the structure deposited by $(X, \Vdash, S)$ on S:

$(S, \mathcal{A}, \mathcal{J})$ with: $\mathcal{A}$ saturation: $U \subseteq \mathcal{A}V \leftrightarrow \mathcal{A}U \subseteq \mathcal{A}V$

$\mathcal{J}$ reduction: $\mathcal{J}U \subseteq V \leftrightarrow \mathcal{J}U \subseteq \mathcal{J}V$

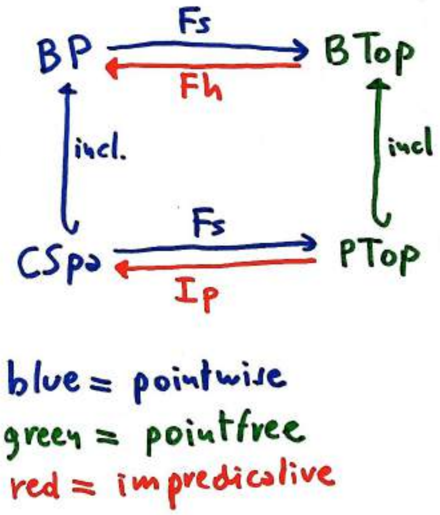
$\mathcal{A}/\mathcal{J}$ compatibility: $\mathcal{A}U \between \mathcal{J}V \leftrightarrow U \between \mathcal{J}V$

$\mathcal{J}$ is **new**: Z formal closed $= \mathcal{J}$-reduced
primitive treatment of closed subsets, also in a pointfree setting

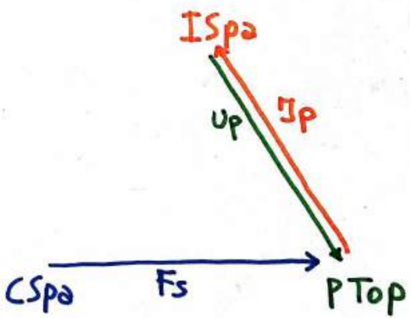
Positive topology: add convergence: $\mathcal{A}U \cap \mathcal{A}V = \mathcal{A}(U{\downarrow}V)$

pointfree topology is more general than topology with points

The well known adjunction between topological spaces **Top** and locales **Loc** becomes an embedding of concrete spaces **CSpa** into positive topologies **PTop**



ideal spaces



for every positive topology $\mathcal{S}$, put $Ip(\mathcal{S}) \equiv (IPt(\mathcal{S}), \ni, \mathcal{S})$ ideal space

for every formal map $s : \mathcal{S} \to \mathcal{T}$, put

$$
\begin{tikzcd}
IPt(\mathcal{S}) \arrow[r, "\ni"] \arrow[d, "s_\exists"'] & \mathcal{S} \arrow[d, "s"'] & \mathcal{S} \arrow[d, "s"] \arrow[dl, phantom, "\leftsquigarrow"] \\
IPt(\mathcal{T}) \arrow[r, "\ni"'] & \mathcal{T} & \mathcal{T}
\end{tikzcd}
$$

ISpa category if ideal spaces = the "image" of **PTop** under *Ip*
Conversely, *Up* forgets all what *Ip* added.

the dark side of the moon

the dark side of the moon = **positive** mathematical treatment of existential statements

the relations $\between, \ltimes, \gtrless, \ldots$ were hidden under classical negation

e.g. $A \between B \leftrightarrow A \cap B \not\subseteq \emptyset$

overlap algebra = locale (frame, cHa) with an extra binary relation $\gtrless$ s.t.

symmetry if $p \gtrless q$ then $q \gtrless p$

preservation of meet: if $p \gtrless q$ then $p \gtrless p \wedge q$

splitting of join: $q \gtrless \bigvee_{i \in I} p_i$ iff $q \gtrless p_i$ for some $i \in I$,

density: $\forall r(p \gtrless r \to q \gtrless r) \to p \leq q$,

A positive treatment of overlap and closure (besides inclusion and interior) makes the language **much more expressive**

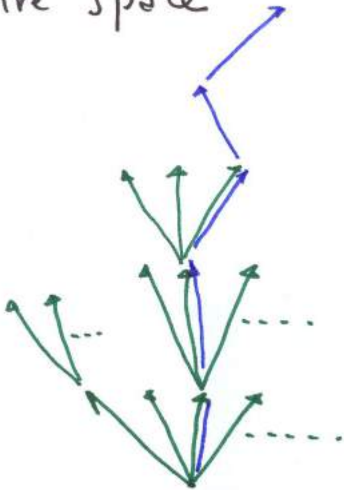
putting topology in algebraic terms

basic notion: overlap algebra + C closure, I interior s.t.

compatibility $Ip \gtrless Cq$ then $Ip \gtrless q$

I-nucleus: $Ip \wedge Iq \leq I(p \wedge q)$

C-I- density: $\forall r(p \gtrless Ir \to q \gtrless Ir) \to p \leq Cq$, Baire space



Baire space

we obtain choice sequences as ideal points over Baire positive topology

$(\mathbb{N}^*, \lhd, \ltimes)$ positive topology

$$\frac{k \epsilon U}{k \lhd U} \qquad \frac{k * N \lhd U}{k \lhd U} \qquad \frac{l \prec k \quad l \lhd U}{k \lhd U}$$

generated by induction

$$\frac{k \ltimes U}{k \epsilon U} \qquad \frac{k \ltimes U}{k * N \ltimes U} \qquad \frac{k \prec l \quad l \ltimes U}{k \ltimes U}$$

generated by coinduction

spread $=$ inhabited subset which is a fixed point for $\ltimes$ (formal closed)

ideal point (in any positive topology) $=$ convergent, formal closed subset

α inhabited, $a, b \epsilon \alpha \rightarrow a{\downarrow}b \between \alpha$, and α is formal closed (hence splits the cover).

Ideal points of Baire positive topology on $\mathbb{N}^*$ coincide with functions from $\mathbb{N}$ to $\mathbb{N}$, i.e. choice sequences.
But we know them to be defined by coinduction, hence we can get every finite initial segment

Baire space

assuming $\texttt{AC!}$ amounts to: every sequence is lawlike

no $\texttt{AC!}$: spatial intuition **can live together with** computational interpretation

notion of choice sequence: very fragile, depends on foundational choices

Bar Induction

$$\texttt{BI} \qquad \forall\alpha(k \epsilon \alpha \to U \between \alpha) \to k \lhd U$$

is just an equivalent formulation of spatiality of Baire positive topology.

The dual to spatiality, reducibility, should be valid: it says that every spread is inhabited by a choice sequence:

$$\texttt{SH} \qquad k \ltimes U \to \exists\alpha(k \epsilon \alpha \ \& \ \alpha \subseteq U)$$

Both $\texttt{BI}$ and $\texttt{SH}$ are perfectly precise and clear mathematical statements. They are intuitively obvious, but they look as unprovable.

Way out: prove meta-mathematically that such ideal principles are **conservative** over real, pointfree topology (work in progress).

Minimalist Foundation

To be able to prove **meta-mathematical** properties:

conservativity of ideal aspects over real mathematics (hopefully),

computational or realizability interpretation, (= concrete interpretation, not just a formal consistency proof)

formalization in a proof-assistant

we need a formal system to be specified in all detail.

the **Minimalist Foundation** MF, as a formal system, is the best we can do, since it respects all ingredients, allows pluralism etc.

It's most important conceptual novelty (paradigm shift) is two levels of abstraction in one formal system.

Conclusions: benefits of a paradigm shift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