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Mathematics as a dynamic process: effects on the working mathematician

XXVI meeting VCA
Padova, Botanical Garden, 25-28 September 2017
The problem of foundations - summary

- non-euclidean geometry: loss of absolute truth in geometry
- abstract algebra (to cope with complexity)
- rigorization of analysis ("pathological curves",...)

- Cantor, Dedekind, Frege, Peano: naive set theory
- Paradoxes, i.e. **contradictions**: Burali-Forti 1896, Russel 1901, ...
- Crisis of foundations
- Traditional ways out:
  - logicism (Frege, Russell, Whitehead, *Principia Mathematica 1911*)
  - constructivism (Kronecker, Borel, Poincaré, Brouwer, Heyting,...)
  - formalism (Hilbert, Zermelo,...)

**Hilbert’s program**: consistency of ZFC, a finitary proof
We don’t have a proof of formal consistency of ZFC, and most probably we will never have one:

\[ \text{ZFC} \nvdash \text{Con}(\text{ZFC}) \] by Gödel’s 2nd incompleteness theorem
Common paradigm today

Somehow paradoxically... the common paradigm is:

Bourbaki’s attitude = denial of the problem
  platonist on weekdays, formalist on sundays  split mind

... when philosophers attack... we rush to hide behind formalism and say
“mathematics is just a combination of meaningless symbols”... we are left in peace...
with the feeling each mathematician has that he is working with something real. This
sensation is probably an illusion, but is very convenient. That is Bourbaki’s attitude
toward foundations.

J. A. Dieudonné, 1970, see Davis-Hersh The mathematical experience, 1981

Formally  classical logic and axiomatic set theory ZFC, ignoring Gödel.
An act of faith remains necessary
Authority and conservativity

One should not have faith in some kind of ideal **authority** from which something follows which is clearly wrong from a more concrete point of view.

**Conservativity** of ideals over real life is missing:

- in religion: slavery, wars, female genital mutilation,...
- in economy: destruction of the planet
- in mathematics: Banach - Tarski, wellordering of real numbers

The reason why non-conservativity is common is that one looks for bivalence, completeness, aiming to achieve safety through **static stability**.
A change of paradigm

Open minded meditation on Göödel’s incompleteness, as opposed to Bourbaki’s dogmatic denial of its significance, leads naturally to conceive mathematics as dynamic, partial, plural, that is, a conquered human achievement, instead of static, complete, unique, that is, a given absolute truth.

In other words, it becomes possible to see mathematics as produced by a Darwinian process of evolution, as all other fields of science.

By the same reason, truth and mathematics cannot be reduced to one system chosen once for all.

Truth and mathematics remain something to conquer, step by step.

Whenever we try to block it, by Göödel’s results it becomes incomplete.
Martin-Löf constructive type theory

full control of information (intensional, computer formalization)
provably consistent (procedure for normalization of proofs)
Dynamic constructivism

Abandoning absolute truth, acts of faith in authority looking for a perspective which allows conservativity leads to a natural, dynamic view.

“It is now generally believed by biologists and by neuroscientists that nothing beyond biology and evolution is theoretically necessary to explain human beings, that is, their bodies and their minds.”

“My general claim is simply that the same holds for logic and mathematics. That is, a naturalistic, evolutionary attitude is fully sufficient, and [...] actually convenient for explaining not only the human body and mind, but also all human intellectual products, including logic and mathematics, which are just the most exact of these. That is, mathematics is a product of our minds and so to explain it we require no more than what is needed by biologists to explain the mind.”

GS, Steps toward a dynamic constructivism, 2003

With hindsight, this is what produces a paradigm shift, in the sense of

Th. Kuhn, The structure of scientific revolutions, 1962, 1970
it is **simpler** and **more effective** to manipulate symbols than things:

```
<table>
<thead>
<tr>
<th>mathematical manipulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>abstraction</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>application</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>reality</td>
</tr>
</tbody>
</table>
```

Mathematics is the exploration of notions and structures of our abstract (reliable) thought which can be useful to understand the world. Every culture has its own mathematics; it is useful to man for survival, a continuation of natural **evolution**.

**some consequences:**

- every notion is the result of human abstraction
- many ways to abstract = many kinds of mathematics, **pluralism**
- application is part of mathematics
- objectivity is a result, not a cause; **dynamic, evolutionary** view of mathematics
Adopting dynamic constructivism means adhering to the principles:

1. *Cultivate pluralism in mathematics and foundations.*
   Different styles in abstraction, which means different foundations, produce different kinds of mathematics and should be respected.

2. *Accept open notions and incomplete theories.*
   The construction of mathematics is a never-ending process and nothing is given in advance
   - no fixed universes of all sets, or of all subsets, or of all propositions.

3. *Preserve all conceptual distinctions (no reductionism).*
   many more primitive notions than usual

4. *Preserve all different levels of abstraction.*
   real/ideal mathematics
   extensional/intensional foundational system
   language/metalanguage
Novelties in mathematics

We get rid of the oppression of an absolute and complete truth, and of some transcendent authority
it’s only our responsibility
We become more relaxed, free and creative
Since nothing is assumed to exist before us, we should be aware in every moment of the abstractions by which we pass from reality to mathematical notions.

**price:** one has to start again **from the beginning:** main task is to find correct constructive definitions

**most interesting, fascinating reward:** several novelties emerge which were **hidden** by stronger foundations, using **PSA, LEM**

**surprise:** the **extra information** which we must keep has a clear logical structure, it is not “code”
Total control of information is not good for the working mathematician.

   e.g. extensional equality of functions

To define subsets and relations we need to forget some piece of information, following the

   forget - restore principle

Each logical constant (connectives and quantifiers) is the reflection at the
(more abstract) object language level of a link between assertions at the (more
concrete) metalevel.

   Principle of reflection

We will see another 2 examples of this (real-ideal, MF with two levels of
abstraction)
duality and symmetry in topology

Management of information means we distinguish two cases:

**Set**: we have effective rules to obtain all elements

**Collection**: we don’t

Hence we do not assume Power Set Axiom PSA.

To produce a topology $\Omega X$ on a set $X$ without PSA it is necessary to start from a base for open subsets $\text{ext}(a) \subseteq X \ (a \in S)$ indexed on a second set $S$.

Equivalently $(X, \models, S)$ where $x \models a \equiv x \in \text{ext}(a)$.

We use the relation **overlap** between subsets $D, E \subseteq X$:

$$E \in D \equiv (\exists x \in X)(x \in E \ \& \ x \in D)$$

Then **interior** and **closure** of $D \subseteq X$ are defined by:

$$x \in \text{int} \ D \equiv \exists a(x \models a \ \& \ \text{ext } a \subseteq D)$$

$$x \in \text{cl} \ D \equiv \forall a(x \models a \rightarrow \text{ext } a \in D)$$

Since $E \in D$ is the logical dual of $E \subseteq D \equiv (\forall x \in X)(x \in E \rightarrow x \in D)$, we find that

**int** and **cl** are defined by **strictly dual formulas**, obtained one from another by swapping $\forall, \exists$ and $\rightarrow, \&$. 
By looking at definitions, one can see that \textit{int} and \textit{cl} are obtained by composing more elementary operators between $\mathcal{PX}$ and $\mathcal{PS}$. Putting:

\[
\begin{align*}
  x \in \text{ext } U &\equiv \Diamond x \uparrow U, & a \in \Diamond D &\equiv \text{ext } a \uparrow D, \\
  x \in \text{rest } U &\equiv \Diamond x \subseteq U, & a \in \Box D &\equiv \text{ext } a \subseteq D,
\end{align*}
\]

then

\[
\text{int} = \text{ext } \Box, \quad \text{cl} = \text{rest } \Diamond
\]

With no conditions on $\vdash$, the structure $(X, \vdash, S)$ is perfectly symmetric. So we define the operators $\mathcal{J}, \mathcal{A}$ on $\mathcal{PS}$ symmetric of \textit{int}, \textit{cl}:

\[
\begin{align*}
  \mathcal{J} &= \Diamond \text{rest}, & \mathcal{A} &= \Box \text{rest}.
\end{align*}
\]

Since \text{ext} $\dashv \Box$ and $\Diamond \dashv \text{rest}$ are adjunctions:

- \textit{int}, $\mathcal{J}$ are reductions (contractive, monotone, idempotent)
- \textit{cl}, $\mathcal{A}$ are saturations (expansive, monotone, idempotent).

Moreover, open subsets of $X$, i.e. fixed points for \textit{int}, coincide with those of the form $\text{ext } U$ for some $U \subseteq S$.

They form a complete lattice which is isomorphic to fixed points for \mathcal{A}, which are hence called formal open subsets.

All this applies dually to closed subsets.
All reduction

\[ \forall X \; \text{set of points, } D \subseteq X \]

concrete open

\[ \text{int } D = D \]

symmetric

\[ \text{ext } U \]

isomorphic

\[ \text{dual} \]

\[ \text{rest } U \]

symmetric

\[ \forall D \; \text{cl } D = D \]

concrete closed

\[ U = \mathcal{A} U \]

formal open

These discoveries were buried under excess of assumptions:

**LEM** forces validity \( \text{cl } = - \text{int } - \)

**PSA** makes the second set \( S \) useless (since \( \Omega X \) itself is a set, and \( \vdash \in \)).

This is sufficient to reject the claim: classical paradigm = “absolute truth”
Convergence as the only mathematical module over fully logical (structural) definitions

Up to here the structure is a basic pair \((X, \sqsubseteq, S)\) where \(\sqsubseteq\) is any relation. Recall that the open subsets are of the form \(\text{ext } U \equiv \bigcup_{b \in U} \text{ext } b\).

A topological space is a set \(X\) of points and a collection of open subsets \(\Omega X\) closed under arbitrary unions and finite intersections

\[
(X, \Omega X) \Rightarrow (X, \epsilon, \Omega X) \Rightarrow (X, \sqsubseteq, S)
\]

where \(\text{ext } a \subseteq X(a \in S)\) is a base for \(\Omega X\) and \(x \sqsubseteq a \equiv x \in \epsilon \text{ ext } a\).

Open subsets form a topology \(\Omega X\) iff \(\text{ext } (a) \subseteq X(a \in S)\) is a base, that is, satisfies convergence:

B1: \(\text{ext } a \cap \text{ext } b = \text{ext } (a \downarrow b)\)

where \(c \in a \downarrow b \equiv \text{ext } c \subseteq \text{ext } a \& \text{ext } c \subseteq \text{ext } b\)

every two neighbourhoods have a common refinement

B2: \(\text{ext } S = X\)

every point has a neighbourhood
Having bases $S, T$, we discover that a function $f : X \to Y$ is continuous from $(X, \Vdash, S)$ into $(Y, \Vdash', T)$ iff there is a relation $s : S \to T$ s.t. $\Vdash' \circ f = s \circ \Vdash$.

By symmetry, it is natural to consider a relation $r$ also between $X$ and $Y$. Then t.f.a.e.:

1. $r$ is continuous, that is $r \times \emptyset \text{ ext } b \to \exists a (x \Vdash a \& \text{ ext } a \subseteq r^- \text{ ext } b)$
2. $r^-$ is open,
3. there exists a relation $s : S \to T$ such that $r^- \text{ ext } b = \text{ ext } s^- b$ for all $b \in T$.

Continuity becomes $\Vdash' \circ r = s \circ \Vdash$, a commutative square of relations:

$$
\begin{array}{c}
X \xrightarrow{\Vdash} S \\
\downarrow r \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow s \\
Y \xrightarrow{\Vdash'} T
\end{array}
$$
categories $\textbf{BP}$ and $\textbf{CSpa}$

$\textbf{BP}$: basic pairs $(X, \vdash, S)$ and relation-pairs $(r, s)$ (commutative squares)

$\textbf{CSpa}$: concrete spaces = convergent basic pairs, that is: B1-B2 hold, or equivalently every $\text{cl}\{x\}$ is convergent

relation-pairs $(r, s)$ preserving convergence:

- $r$ maps convergent subsets into convergent subsets
- convergent subset = topological notion of point
- $r^-$ respects finite intersections
- $r^- \text{ ext } (b \downarrow_Y c) = \text{ ext } (s^- b \downarrow_X s^- c)$, for all $b, c \in T$, and $r^- \text{ ext } T = \text{ ext } S$,
Pointfree topology

The pointfree approach to topology is a must. In many cases, points do not form a set. So we must obtain them as ideal points over an effective, pointfree structure (exactly as: real numbers are ideal points over unions of intervals with rational end-points)

general methodology: axiomatise the structure induced on the formal side

basic pair \( \mathcal{X} \) ⇒ basic topology \( Fs(\mathcal{X}) \)

\((r, s): \mathcal{X} \rightarrow \mathcal{Y}\) relation-pair ⇒ \(s\) continuous relation from \(Fs(\mathcal{X}) \rightarrow Fs(\mathcal{Y})\)
in general, \(s: S \rightarrow T\) if \(s\) preserves \(\mathcal{J}\) and \(s - \ast\) preserves \(\mathcal{A}\)

concrete space ⇒ positive topology

\((r, s): \mathcal{X} \rightarrow \mathcal{Y}\) convergent relation-pair

⇒ \(s\) formal map \(Fs(\mathcal{X}) \rightarrow Fs(\mathcal{Y})\)

singleton \(\{x\}\), or equivalently convergent subset \(D\)

⇒ \(\diamond x\) and \(\diamond D\) ideal point of \(Fs(\mathcal{X})\)
Basic topology: axiomatization of the structure deposited by \((X, \models, S)\) on \(S\):

\((S, \mathcal{A}, \mathcal{J})\) with:
- \(\mathcal{A}\) saturation: \(U \subseteq AV \iff AU \subseteq AV\)
- \(\mathcal{J}\) reduction: \(JU \subseteq V \iff JU \subseteq JV\)
- \(\mathcal{A}/\mathcal{J}\) compatibility: \(AU \upharpoonright JV \iff U \upharpoonright JV\)

\(\mathcal{J}\) is new: \(Z\) formal closed = \(\mathcal{J}\)-reduced
primitive treatment of closed subsets, also in a pointfree setting

Positive topology: add convergence: \(AU \cap AV = A(U \upharpoonright V)\)
pointfree topology is more general than topology with points.

The well known adjunction between topological spaces $\text{Top}$ and locales $\text{Loc}$ becomes an embedding of concrete spaces $\text{CSpa}$ into positive topologies $\text{PTop}$.

\[
\begin{array}{ccc}
\text{BP} & \xrightarrow{\text{Fs}} & \text{BTop} \\
\downarrow \text{incl.} & & \downarrow \text{incl} \\
\text{CSpa} & \xrightarrow{\text{Fs}} & \text{PTop} \\
\end{array}
\]

\text{blue} = \text{pointwise} \\
\text{green} = \text{pointfree} \\
\text{red} = \text{impredicative}
ideal spaces

for every positive topology $S$, put $lp(S) \equiv (lPt(S), \exists, S)$.

for every formal map $s : S \to T$, put $lPt(S) \xrightarrow{s} S$.

$I_{Spa}$ category if ideal spaces $= \text{the “image” of } PT_{op} \text{ under } lp$.

Conversely, $Up$ forgets all what $lp$ added.
the dark side of the moon

the dark side of the moon = positive mathematical treatment of existential statements

the relations $\square, \times, \equiv,...$ were hidden under classical negation
e.g. $A \parallel B \iff A \cap B \not\subseteq \emptyset$

overlap algebra = locale (frame, cHa) with an extra binary relation $\equiv$ s.t.

symmetry \quad if $p \equiv q$ then $q \equiv p$

preservation of meet: \quad if $p \equiv q$ then $p \equiv p \land q$

splitting of join: \quad $q \equiv \bigvee_{i \in I} p_i$ \iff $q \equiv p_i$ for some $i \in I$,

density: \quad $\forall r(p \equiv r \rightarrow q \equiv r) \rightarrow p \leq q,$

A positive treatment of overlap and closure (besides inclusion and interior) makes the language much more expressive

putting topology in algebraic terms

basic notion: overlap algebra $+$ $C$ closure, $I$ interior s.t.

compatibility \quad $lp \equiv Cq$ then $lp \equiv q$

$I$-nucleus: \quad $lp \land lq \leq l(p \land q)$

$C$-$I$- density: \quad $\forall r(p \equiv lr \rightarrow q \equiv lr) \rightarrow p \leq Cq,$
Baire space
we obtain choice sequences as ideal points over Baire positive topology

\((\mathbb{N}^*, \triangleleft, \nabla)\) positive topology

\[
\begin{array}{c}
\frac{k \in U}{k \triangleleft U} & \frac{k \ast N \triangleleft U}{k \triangleleft U} & \frac{l \nabla k \triangleleft U}{k \triangleleft U} \\
\end{array}
\]
generated by induction

\[
\begin{array}{c}
\frac{k \nabla U}{k \epsilon U} & \frac{k \ast U \nabla U}{k \nabla U} & \frac{k \nabla l \nabla U}{k \nabla U} \\
\end{array}
\]
generated by coinduction

spread = inhabited subset which is a fixed point for \(\nabla\) (formal closed)

ideal point (in any positive topology) = convergent, formal closed subset

\(\alpha\) inhabited, \(a, b \in \alpha \rightarrow a \downarrow b \nsubseteq \alpha\), and \(\alpha\) is formal closed (hence splits the cover).

Ideal points of Baire positive topology on \(\mathbb{N}^*\) coincide with functions from \(\mathbb{N}\) to \(\mathbb{N}\), i.e. choice sequences.

But we know them to be defined by coinduction, hence we can get every finite initial initial segment
Baire space

assuming AC! amounts to: every sequence is lawlike

no AC!: spatial intuition can live together with computational interpretation

notion of choice sequence: very fragile, depends on foundational choices

Bar Induction

\[
\text{BI} \quad \forall \alpha (k \in \alpha \rightarrow U \nsubseteq \alpha) \rightarrow k < U
\]

is just an equivalent formulation of spatiality of Baire positive topology.

The dual to spatiality, reducibility, should be valid: it says that every spread is inhabited by a choice sequence:

\[
\text{SH} \quad k \times U \rightarrow \exists \alpha (k \in \alpha \text{ & } \alpha \subseteq U)
\]

Both BI and SH are perfectly precise and clear mathematical statements. They are intuitively obvious, but they look as unprovable.

Way out: prove meta-mathematically that such ideal principles are conservative over real, pointfree topology (work in progress).
To be able to prove *meta-mathematical* properties:
   - conservativity of ideal aspects over real mathematics (hopefully),
   - computational or realizability interpretation, (＝ concrete interpretation, not just a formal consistency proof )
   - formalization in a proof-assistant
we need a formal system to be specified in all detail.

the **Minimalist Foundation** MF, as a formal system, is the best we can do, since it respects all ingredients, allows pluralism etc.

It’s most important conceptual novelty (paradigm shift) is two levels of abstraction in one formal system.
Conclusions: benefits of a paradigm shift