Useful axioms

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Non-constructive principles for mathematics

A list of five (in some cases apparently unrelated) useful non-constructive principles:

1. The axiom of choice,
2. Baire's category theorem,
3. Large cardinal axioms,
4. Shoenfield's absoluteness,
5. Łoś Theorem for ultrapowers of first orders structures.

First aim: show that the language of forcing allows to bring out the analogies more or less evident between all these distinct principles and to express all of them as forcing axioms.

Second aim: formulate stronger and stronger non constructive principles leveraging on different aspects of the above analogies.

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The axiom of choice is a global forcing axiom

This observation has been handled to me by Stevo Todorčević.

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Definition

Let λ be an infinite cardinal. DC_λ holds if for all sets X and all functions $F : X^{<\lambda} \to P(X)$, there exists $g : \lambda \to X$ such that $g(\alpha) \in F(g \restriction \alpha)$ for all $\alpha < \lambda$.

Fact

The axiom of choice AC *is equivalent over* ZF *to the assertion* DC_λ *holds for all* λ.

This is a local statement, i.e. there is a level by level correspondance between the amount of choice and of dependent choice available in the universe.

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Let P be a partial order. $\mathsf{FA}_\lambda(P)$ holds if for all family $\{D_\alpha : \alpha < \lambda\}$ of dense subsets of P, there exists a filter $G \subset P$ which has non-empty intersection with all the D_α.

Let Γ be a class of partial orders. Then $\mathsf{FA}_\lambda(\Gamma)$ holds if $\mathsf{FA}_\lambda(P)$ holds for all $P \in \Gamma$.

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$\mathsf{DC}_{\aleph_0}$ *is equivalent over* ZF *to the assertion* $\mathsf{FA}_{\aleph_0}(P)$ *holds for all* P.

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Sketch of proof. I show just the direction I want to bring forward: Assume $F : X^{<\omega} \to P(X)$ is a function. Let T be the subtree of $X^{<\omega}$ given by finite sequences $s \in X^{<\omega}$ such that $s(i) \in F(s \restriction i)$ for all $i < |s|$.
Consider the family given by the dense sets

$$D_n = \{s \in T : |s| > n\}.$$

If G is a filter on T meeting the dense sets of this family, $\cup G$ works.

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More generally:

Definition

A partial order P is $< \lambda$-closed if all chains in P of length less than λ have a lower bound.

Let $\mathsf{AC} \restriction \lambda$ abbreviate DC_γ *holds for all* $\gamma < \lambda$ and Γ_λ be the class of $< \lambda$-closed posets.

Fact

DC_λ *is equivalent to* $\mathsf{FA}_\lambda(\Gamma_\lambda)$ *over the theory* $\mathsf{ZF} + \mathsf{AC} \restriction \lambda$.

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Conclusion:

Fact

The axiom of choice is equivalent over the theory ZF *to the assertion that* $\mathsf{FA}_\lambda(\Gamma_\lambda)$ *holds for all* λ.

The usual forcing axioms such as Martin's maximum or the proper forcing axiom are natural strenghtenings of the axiom of choice. They aim to isolate a maximal strengthening of $\mathsf{AC} \restriction \omega_2$ enlarging the family Γ for which $\mathsf{FA}_{\aleph_1}(\Gamma)$ holds.

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Theorem (BCT)

Assume (X,τ) is compact and Hausdorff. Let $\{D_n : n \in \omega\}$ be a family of dense open subsets of X. Then $\bigcap_{n\in\omega} D_n$ is non-empty.

Fact

$\mathrm{FA}_{\aleph_0}(P)$ *for all forcing P entails* BCT.

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Proof.

Let (X, τ) compact Hausdorff and $\{D_n : n \in \omega\}$ a family of dense open subsets of X.
Let $(P, \leq_P) = (\tau \setminus \{\emptyset\}, \subseteq)$ and

$$E_n = \{A \in \tau : \overline{A} \subseteq D_n\}.$$

Each E_n is a dense subset of P. Let G be a filter on P with $G \cap E_n \neq \emptyset$ for all n. By compactness of X

$$\emptyset \neq \bigcap \{\mathrm{Cl}\,(A) : A \in G\} \subseteq \bigcap_{n \in \omega} D_n.$$

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More general forcing axioms

Fact

Let G be a filter on a poset P and $X \subseteq P$. Then $G \cap X$ is non-empty iff $G \cap \downarrow X$ is non-empty.

Hence G meets a predense set A iff it meets the dense open set $\downarrow A$.

Definition

Given a poset P and a property ϕ, $\mathrm{FA}_\phi(P)$ holds if

> *For all $\mathcal{D}$ collection of predense subsets of P such that $\phi(\mathcal{D})$ holds, there exists G filter on P such that $G \cap X \neq \emptyset$ for all $X \in \mathcal{D}$.*

$\mathrm{FA}_\kappa(P)$ stands for $\mathrm{FA}_\phi(P)$ where
$\phi(\mathcal{D}) \equiv |\mathcal{D}| = \kappa$ *and each* $D \in \mathcal{D}$ *is predense.*
$\mathrm{BFA}_{\omega_1}(P)$ stands for $\mathrm{FA}_\phi(\mathrm{RO}(P))$ where
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Large cardinals as forcing axioms

Given a cardinal κ,

- I_κ is the ideal of bounded subsets of κ,
- $\mathcal{A}_\kappa$ is the family of maximal antichains of size less than κ in $\mathcal{P}(\kappa)/_{I_\kappa}$.

Definition

κ is measurable iff there is a ultrafilter $G \in \mathcal{P}(\kappa)/_{I_\kappa}$ such that $G \cap A \neq \emptyset$ for all $A \in \mathcal{A}_\kappa$.

I.e. κ is measurable if $\mathsf{FA}_\phi(\mathcal{P}(\kappa)/_{I_\kappa})$, where $\phi(\mathcal{D})$ stands for $\mathcal{D} = \mathcal{A}_\kappa$.
Cofinally many large cardinal properties of κ can be formulated as forcing axiom of the type $\mathsf{FA}_\phi(\mathcal{P}(\mathcal{P}(\lambda))/_{J_{\kappa,\lambda}})$, choosing ϕ and $J_{\kappa,\lambda}$ suitably.
for example supercompact, huge, extendible, n-huge, I_1, etc......

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Łoś theorem

Theorem

Let $\{\mathfrak{M}_l = (M_l, R_l) : l \in L\}$ be first oreder models for $\mathcal{L} = \{R\}$. Let $G \subseteq \mathcal{P}(L)$ be a ultrafilter on L. Set

- $[f]_G = [h]_G$ *iff* $\{l \in L : f(l) = h(l))\} \in G$,
- $\bar{R}([f_1]_G, \ldots, [f_n]_G)$ *iff* $\{l \in L : R_l(f_1(l), \ldots, f_n(l))\} \in G$.

Then:

1. *For all* $\phi(x_1, \ldots, x_n)$ $(\prod_{l \in L} M_l/G, \bar{R}) \models \phi([f_1]_G, \ldots, [f_n]_G)$ *if and only if* $\{l \in L : \mathfrak{M}_l \models \phi(f_1(l), \ldots, f_n(l))\} \in G$.
2. *If* $\mathfrak{M}_l = \mathfrak{M}$ *for all* $l \in L$, $M \prec \prod_{l \in L} M_l/G$ *as witnessed by the map* $m \mapsto [c_m]_G$ *(where* $c_m : L \to M$ *is constant with value* m*).*

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Recall on boolean algebras and Stone spaces

Given a boolean algebra B:

- St(B) is given by its ultrafilters G.
- St(B) is endowed with a *compact, Hausdorff* topology τ_{B} whose clopens are $N_b = \{G \in \mathrm{St(B)} : b \in G\}$.
- The map $b \mapsto N_b$ defines a natural isomorphism of B with the boolean algebra CLOP(St(B)) of clopen subset of St(B).

- B is *complete* if and only if $\mathrm{CLOP(St(B))} = \mathrm{RO}(\mathrm{St(B)}, \tau_{\mathrm{B}}) \cong \mathrm{B}$.
- Spaces X satisfying the property that its regular open sets are closed are *extremally (or extremely) disconnected.*
- $\mathcal{P}(X)$ is a complete boolean algebra, and $\beta(X) = \mathrm{St}(\mathcal{P}(X))$ is the Stone-Cech compactification of X with discrete topology and is extremally disconnected.

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- B is *complete* if and only if $\mathrm{CLOP}(\mathrm{St}(\mathrm{B})) = \mathrm{RO}(\mathrm{St}(\mathrm{B}), \tau_{\mathrm{B}}) \cong \mathrm{B}$.
- Spaces X satisfying the property that its regular open sets are closed are *extremally (or extremely) disconnected*.
- $\mathcal{P}(X)$ is a complete boolean algebra, and $\beta(X) = \mathrm{St}(\mathcal{P}(X))$ is the Stone-Cech compactification of X with discrete topology and is extremally disconnected.

Recall on boolean algebras and Stone spaces

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Boolean valued models

Definition

Let B be a *cba* and a $\mathcal{L}$ be first order *relational* language.
A B-*valued model* for $\mathcal{L}$ is a tuple
$\mathcal{M} = \langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, c_j^{\mathcal{M}} : j \in J\rangle$ with

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\begin{aligned}
=^{\mathcal{M}}\colon M^2 &\to \mathsf{B} \\
(\tau,\sigma) &\mapsto [\![\tau = \sigma]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![\tau = \sigma]\!]\,,
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R^{\mathcal{M}} : M^n &\to \mathsf{B} \\
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Forcing relations on boolean valued models

The constraints on $R^{\mathcal{M}}$ and $=^{\mathcal{M}}$ are the following:

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Boolean valued semantics

Definition

Let $\langle M, =^{\mathcal{M}}, R^{\mathcal{M}}\rangle$ be a B-valued model in the relational language $\mathcal{L} = \{R\}$, $\phi(x_1, \ldots, x_n)$ a $\mathcal{L}$-formula with displayed free variables, ν : free variables $\to M$.

$[\![\phi]\!]_{\mathsf{B}}^{\mathcal{M},\nu} = [\![\phi]\!]$, the *boolean value* of ϕ with the assignment ν is defined by recursion as follows:

- $[\![t = s]\!] = [\![\nu(t) = \nu(s)]\!]$,
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Soundness Theorem for B-valued semantics

Theorem (Soundness Theorem)

Assume $\mathcal{L}$ is a relational language and ϕ is a $\mathcal{L}$-formula which is syntactically provable by a $\mathcal{L}$-theory T.
Assume each formula in T has boolean value at least $b \in \mathsf{B}$ in a B-valued model $\mathcal{M}$ with valuation ν.
Then $[\![\phi]\!]_{\mathsf{B}}^{\mathcal{M},\nu} \geq b$ as well.

The completeness theorem is automatic given that 2 is a complete boolean algebra.

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Tarski quotient of B-valued models

Definition

Let B be a *cba*, $\mathcal{M}$ a B-valued model for $\mathcal{L}$, and G a ultrafilter over B. Consider the following equivalence relation on M:

$$\tau \equiv_G \sigma \Leftrightarrow [\![\tau = \sigma]\!] \in G$$

The first order (Tarski) model $\mathcal{M}/_G = \langle M/_G, R_i^{\mathcal{M}/G} : i \in I, c_j^{\mathcal{M}/G} : j \in J \rangle$ is defined letting:

- For any n-ary relation symbol R in $\mathcal{L}$

$$R^{\mathcal{M}/G} = \{([\tau_1]_G, \ldots, [\tau_n]_G) \in (M/_G)^n : [\![R(\tau_1, \ldots, \tau_n)]\!] \in G\}.$$

- For any constant symbol c in $\mathcal{L}$

$$c^{\mathcal{M}/G} = [c^{\mathcal{M}}]_G.$$

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Full B-valued models

Definition

A B-valued model $\mathcal{M}$ for the language $\mathcal{L}$ is *full* if for every $\mathcal{L}$-formula $\phi(x, \bar{y})$ and every $\bar{\tau} \in M^{|\bar{y}|}$ there is a $\sigma \in M$ such that

$$[\![\exists x \phi(x, \bar{\tau})]\!] = [\![\phi(\sigma, \bar{\tau})]\!]$$

Boolean valued Łoś Theorem — Forcing theorem

Theorem (B-valued Łoś's Theorem — Forcing theorem)

Assume $\mathcal{M}$ is a full B*-valued model for the relational language $\mathcal{L}$. Then for every formula $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}$ and $(\tau_1, \ldots, \tau_n) \in M^n$:*

1. *For all ultrafilters G over* B*, $\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$ if and only if $[\![\phi(\tau_1, \ldots, \tau_n)]\!] \in G$.*
2. *For all $a \in$ B the following are equivalent:*
    1. $[\![\phi(f_1, \ldots, f_n)]\!] \geq a$,
    2. $\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$ *for all* $G \in N_a$,
    3. $\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$ *for densely many* $G \in N_a$.

Łoś's Theorem versus boolean valued Łoś's Theorem

Fact

Let $(M_x : x \in X)$ be a family of Tarski-models in the first order relational language $\mathcal{L}$. Then $N = \prod_{x \in X} M_x$ is a full $\mathcal{P}(X)$-model, letting for each n-ary relation symbol $R \in \mathcal{L}$,
$[\![R(f_1, \ldots, f_n)]\!]_{\mathcal{P}(X)} = \{x \in X : M_x \models R(f_1(x), \ldots, f_n(x))\}$.

Let G be any non-principal ultrafilter on X. Then the Tarski quotient N/G is the familiar ultraproduct of the family $(M_x : x \in X)$ by G.

> *The usual Łoś theorem for ultraproducts of Tarski models is the specialization to the case of the full $\mathcal{P}(X)$-valued model N of the boolean valued Łoś theorem.*
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> *If N is an ultrapower of a model M, the embedding $a \mapsto [c_a]_G$ (where $c_a(x) = a$ for all $x \in X$ and $a \in M$) is elementary.*

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Boolean ultrapowers of compact Hausdorff spaces

Let X be a set with the discrete topology.

- For $a \in X$, $G_a \in \mathrm{St}(\mathcal{P}(X))$ is the principal ultrafilter of supersets of $\{a\}$.
- The map $a \mapsto G_a$ embeds X as an open, dense, discrete subspace of $\mathrm{St}(\mathcal{P}(X))$.
- For any space (Y, τ), any $f : X \to Y$ is continuous. (since X has the discrete topology)

Moreover if Y is compact Hausdorff:

- $f : X \to Y$ induces a unique continuous extension $\bar{f} : \mathrm{St}(\mathcal{P}(X)) \to Y$. ($\mathrm{St}(\mathcal{P}(X))$ is also the Stone-Cech compactification of X).
- $C(X, Y) = Y^X$ is canonically isomorphic to $C(\mathrm{St}(\mathcal{P}(X)), Y)$.
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Boolean ultrapowers of 2^ω

Let B be an arbitrary complete boolean algebra, and set $M = C(St(\mathsf{B}), 2^\omega)$.

Fix R a Borel (Universally Baire) relation on $(2^\omega)^n$. The continuity of an n-tuple $f_1, \ldots, f_n \in M$ grants that

$$\{G : R(f_1(G) \ldots, f_n(G))\} = (f_1 \times \cdots \times f_n)^{-1}[R]$$

has the Baire property in St(B), where $f_1 \times \cdots \times f_n(G) = (f_1(G), \ldots, f_n(G))$.
Define:

$$\begin{aligned} R^M :& M^n \to \mathsf{B} \\ & (f_1, \ldots, f_n) \mapsto \mathrm{Reg}\left(\{G : R(f_1(G), \ldots, f_n(G)\}\right) \end{aligned}$$

where $\mathrm{Reg}\,(A) = \mathrm{Int}\,(\mathrm{Cl}\,(A))$.
Also, since the diagonal is closed in $(2^\omega)^2$,

$$=^M (f, g) = \mathrm{Reg}\left(\{G : f(G) = g(G)\}\right)$$

is well defined.

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where $\mathrm{Reg}\,(A) = \mathrm{Int}\,(\mathrm{Cl}\,(A))$.

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Let B be an arbitrary (even atomless) complete boolean algebra. The following holds:

- For any Borel (universally Baire) relation R on $(2^\omega)^n$, the structure $(M, =^M, R^M)$ is a *full* B-valued model.
- For $G \in \mathrm{St}(\mathsf{B})$,

$$\begin{aligned} i_G : 2^\omega &\to M/G \\ x &\mapsto [c_x]_G \end{aligned}$$

  (c_x is the constant function with value x) defines an injective morphism $(2^\omega, R)$ into $(M/G, R^M/G)$.

It is not clear whether this morphism is an elementary map or not:

- This is the case for $\mathsf{B} = \mathcal{P}(X)$, since in this case we are analyzing the standard embedding of the first order structure $(2^\omega, R)$ in its ultrapowers induced by ultrafilters on $\mathcal{P}(X)$.
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Shoenfield's absoluteness rephrased

Theorem (Cohen's absoluteness)

Assume B *is a complete boolean algebra and* $R \subseteq (2^\omega)^n$ *is a Borel (Universally Baire) relation. Let* $M = C(St(\mathsf{B}), 2^\omega)$ *and* $G \in \mathrm{St}(\mathsf{B})$. *Then*

$$(2^\omega, =, R) \prec_{\Sigma_2} (M/_G, =^M/_G, R^M/_G).$$

If one assumes the existence of class many Woodin cardinals

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Proof.

$C(St(\mathsf{B}), 2^\omega)$ is isomorphic to the B-names in V^{B} for elements of 2^ω (see next slide). Apply Shoenfield's (or Woodin's) absoluteness to V and $V[H]$ (for H V-generic for B) to infer the desired conclusion. □

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$C(\mathrm{St}(\mathsf{B}), 2^\omega)$ and V^{B}

Given $f \in C(\mathrm{St}(\mathsf{B}), 2^\omega) = M$, $\sigma \in V^{\mathsf{B}}$ with $[\![\sigma \in 2^\omega]\!] = 1_{\mathsf{B}}$ define:

- $\tau_f = \left\{\langle\langle n, i\rangle, f^{-1}[N_{n,i}]\rangle : n < \omega, i < 2\right\} \in V^{\mathsf{B}}$,
- $g_\sigma \in M$ by $g_\sigma(G)(n) = i$ iff $[\![\sigma(n) = i]\!] \in G$.

Then

- $g_{\tau_f} = f$,
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These identities allow to translate forcing relations from both sides.

The lift of a Universally Baire relation R to V^{B} is translated as the forcing relation (on M)

$$\begin{aligned} R^M : & M^n \to \mathsf{B} \\ & (f_1, \ldots, f_n) \mapsto \mathrm{Reg}\left(\{G : R(f_1(G), \ldots, f_n(G))\}\right). \end{aligned}$$

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Comments and open questions

- Forcing axioms spring out from a natural inquire to strengthen as much as possible the nonconstructive tools.
- Most often BCT and AC suffice. In some cases (which are not restricted to set theory but occurs also in other parts of mathematics) generic absoluteness arguments for projective sets are useful.
- It is also the case that generic absoluteness results for third order number theory spring out as natural consequences of strong forcing axioms. A theme I do not have time to expand on here.
- There is a whole lot to say on the possibility to use forcing axioms to decide third order aritmetic much in the same way as Woodin and Shoenfield's absoluteness do for second order number theory. We refer the reader to the bibliography for further informations.

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