CLASSICAL LIE THEORY FROM THE POINT OF VIEW OF MONADS

Based on [A. Ardizzoni, J. Gómez-Torrecillas and C. Menini, *Monadic Decompositions and Classical Lie Theory*, Appl. Categor. Struct., Online First.]

Alessandro Ardizzoni*, José Gómez-Torrecillas and Claudia Menini

ALGEBRAIC STRUCTURES AND THEIR APPLICATIONS
with a day dedicated to Alberto Facchini on the occasion of his 60th birthday

June 16-20 2014, Spineto (Siena)

Definition

Recall that a monad on a category $\mathscr{C}$ is a triple $\mathbb{Q} := (Q, m, u)$, where

Definition

Recall that a monad on a category $\mathscr{C}$ is a triple $\mathbb{Q} := (Q, m, u)$, where

- $Q : \mathscr{C} \to \mathscr{C}$ is a functor,

Definition

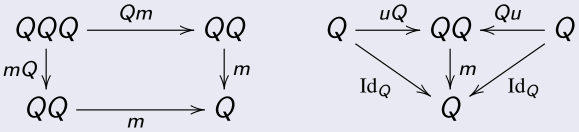
Recall that a monad on a category $\mathscr{C}$ is a triple $\mathbb{Q} := (Q, m, u)$, where

- $Q : \mathscr{C} \to \mathscr{C}$ is a functor,
- $m : QQ \to Q$ and $u : \mathrm{Id}_{\mathscr{C}} \to Q$ are functorial morphisms s.t.

Definition

Recall that a monad on a category $\mathscr{C}$ is a triple $\mathbb{Q} := (Q, m, u)$, where

- $Q : \mathscr{C} \to \mathscr{C}$ is a functor,
- $m : QQ \to Q$ and $u : \mathrm{Id}_{\mathscr{C}} \to Q$ are functorial morphisms s.t.



Definition

Recall that a monad on a category $\mathscr{C}$ is a triple $\mathbb{Q} := (Q, m, u)$, where

- $Q : \mathscr{C} \to \mathscr{C}$ is a functor,
- $m : QQ \to Q$ and $u : \mathrm{Id}_{\mathscr{C}} \to Q$ are functorial morphisms s.t.

$$\begin{array}{ccc} QQQ & \xrightarrow{Qm} & QQ \\ {\scriptstyle mQ}\downarrow & & \downarrow{\scriptstyle m} \\ QQ & \xrightarrow[m]{} & Q \end{array} \qquad \begin{array}{ccccc} Q & \xrightarrow{uQ} & QQ & \xleftarrow{Qu} & Q \\ & {\scriptstyle \mathrm{Id}_Q}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle \mathrm{Id}_Q} & \\ & & Q & & \end{array}$$

A $\mathbb{Q}$-algebra is a pair (X, μ) where $X \in \mathscr{C}$ and $\mu : QX \to X$ is a morphism in $\mathscr{C}$ s.t.

Definition

Recall that a monad on a category $\mathscr{C}$ is a triple $\mathbb{Q} := (Q, m, u)$, where

- $Q : \mathscr{C} \to \mathscr{C}$ is a functor,
- $m : QQ \to Q$ and $u : \mathrm{Id}_{\mathscr{C}} \to Q$ are functorial morphisms s.t.

$$\begin{array}{ccc} QQQ & \xrightarrow{Qm} & QQ \\ {\scriptstyle mQ}\downarrow & & \downarrow{\scriptstyle m} \\ QQ & \xrightarrow[m]{} & Q \end{array} \qquad \begin{array}{ccccc} Q & \xrightarrow{uQ} & QQ & \xleftarrow{Qu} & Q \\ & {\scriptstyle \mathrm{Id}_Q}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle \mathrm{Id}_Q} & \\ & & Q & & \end{array}$$

A $\mathbb{Q}$-algebra is a pair (X, μ) where $X \in \mathscr{C}$ and $\mu : QX \to X$ is a morphism in $\mathscr{C}$ s.t.

$$\begin{array}{ccc} QQX & \xrightarrow{Q\mu} & QX \\ {\scriptstyle mX}\downarrow & & \downarrow{\scriptstyle \mu} \\ QX & \xrightarrow[\mu]{} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{uX} & QX \\ & {\scriptstyle \mathrm{Id}_X}\searrow & \downarrow{\scriptstyle \mu} \\ & & X \end{array}$$

Definition

Recall that a monad on a category $\mathscr{C}$ is a triple $\mathbb{Q} := (Q, m, u)$, where

- $Q : \mathscr{C} \to \mathscr{C}$ is a functor,
- $m : QQ \to Q$ and $u : \mathrm{Id}_{\mathscr{C}} \to Q$ are functorial morphisms s.t.

$$\begin{array}{ccc} QQQ & \xrightarrow{Qm} & QQ \\ {\scriptstyle mQ}\downarrow & & \downarrow{\scriptstyle m} \\ QQ & \xrightarrow[m]{} & Q \end{array} \qquad \begin{array}{ccccc} Q & \xrightarrow{uQ} & QQ & \xleftarrow{Qu} & Q \\ & {\scriptstyle \mathrm{Id}_Q}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle \mathrm{Id}_Q} & \\ & & Q & & \end{array}$$

A $\mathbb{Q}$-algebra is a pair (X, μ) where $X \in \mathscr{C}$ and $\mu : QX \to X$ is a morphism in $\mathscr{C}$ s.t.

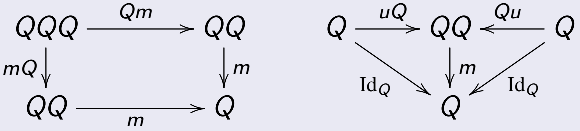
$$\begin{array}{ccc} QQX & \xrightarrow{Q\mu} & QX \\ {\scriptstyle mX}\downarrow & & \downarrow{\scriptstyle \mu} \\ QX & \xrightarrow[\mu]{} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{uX} & QX \\ & {\scriptstyle \mathrm{Id}_X}\searrow & \downarrow{\scriptstyle \mu} \\ & & X \end{array}$$

$\mathbb{Q}$-algebras and their morphisms form the so-called Eilenberg-Moore category ${}_{\mathbb{Q}}\mathscr{C}$ of the monad $\mathbb{Q}$.

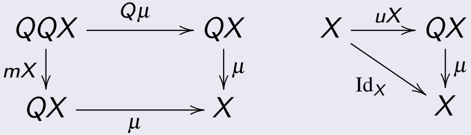
Definition

Recall that a monad on a category $\mathscr{C}$ is a triple $\mathbb{Q} := (Q, m, u)$, where

- $Q : \mathscr{C} \to \mathscr{C}$ is a functor,
- $m : QQ \to Q$ and $u : \mathrm{Id}_{\mathscr{C}} \to Q$ are functorial morphisms s.t.



A $\mathbb{Q}$-algebra is a pair (X, μ) where $X \in \mathscr{C}$ and $\mu : QX \to X$ is a morphism in $\mathscr{C}$ s.t.



$\mathbb{Q}$-algebras and their morphisms form the so-called Eilenberg-Moore category ${}_{\mathbb{Q}}\mathscr{C}$ of the monad $\mathbb{Q}$. When the multiplication and unit of the monad are clear from the context, we will just write Q instead of $\mathbb{Q}$.

A monad $\mathbb{Q}$ on $\mathscr{C}$ gives rise to an adjunction $(F,U) := ({}_{\mathbb{Q}}F, {}_{\mathbb{Q}}U)$ where $U : {}_{\mathbb{Q}}\mathscr{C} \to \mathscr{C}$ is the *forgetful functor* and $F : \mathscr{C} \to {}_{\mathbb{Q}}\mathscr{C}$ is the *free functor*.

A monad $\mathbb{Q}$ on $\mathscr{C}$ gives rise to an adjunction $(F,U) := ({}_{\mathbb{Q}}F, {}_{\mathbb{Q}}U)$ where $U : {}_{\mathbb{Q}}\mathscr{C} \to \mathscr{C}$ is the *forgetful functor* and $F : \mathscr{C} \to {}_{\mathbb{Q}}\mathscr{C}$ is the *free functor*. Explicitly:

$$U(X,\mu) := X, \quad Uf := f \qquad \text{and} \qquad FX := (QX, mX), \quad Ff := Qf.$$

A monad $\mathbb{Q}$ on $\mathscr{C}$ gives rise to an adjunction $(F,U) := ({}_{\mathbb{Q}}F, {}_{\mathbb{Q}}U)$ where $U : {}_{\mathbb{Q}}\mathscr{C} \to \mathscr{C}$ is the *forgetful functor* and $F : \mathscr{C} \to {}_{\mathbb{Q}}\mathscr{C}$ is the *free functor*. Explicitly:

$$U(X,\mu) := X, \quad Uf := f \qquad \text{and} \qquad FX := (QX, mX), \quad Ff := Qf.$$

Note that

- $UF = Q$.

A monad $\mathbb{Q}$ on $\mathscr{C}$ gives rise to an adjunction $(F,U) := ({}_{\mathbb{Q}}F, {}_{\mathbb{Q}}U)$ where $U : {}_{\mathbb{Q}}\mathscr{C} \to \mathscr{C}$ is the *forgetful functor* and $F : \mathscr{C} \to {}_{\mathbb{Q}}\mathscr{C}$ is the *free functor*. Explicitly:

$$U(X,\mu) := X, \quad Uf := f \qquad \text{and} \qquad FX := (QX, mX), \quad Ff := Qf.$$

Note that

- $UF = Q$.
- The unit of the adjunction (F,U) is given by the unit $u : \mathrm{Id}_{\mathscr{C}} \to UF = Q$ of the monad $\mathbb{Q}$.

A monad $\mathbb{Q}$ on $\mathscr{C}$ gives rise to an adjunction $(F,U) := ({}_{\mathbb{Q}}F, {}_{\mathbb{Q}}U)$ where $U : {}_{\mathbb{Q}}\mathscr{C} \to \mathscr{C}$ is the *forgetful functor* and $F : \mathscr{C} \to {}_{\mathbb{Q}}\mathscr{C}$ is the *free functor*. Explicitly:

$$U(X,\mu) := X, \quad Uf := f \qquad \text{and} \qquad FX := (QX, mX), \quad Ff := Qf.$$

Note that

- $UF = Q$.
- The unit of the adjunction (F,U) is given by the unit $u : \mathrm{Id}_{\mathscr{C}} \to UF = Q$ of the monad $\mathbb{Q}$.
- The counit $\lambda : FU \to \mathrm{Id}_{{}_{\mathbb{Q}}\mathscr{C}}$ is uniquely determined by the equality $U\lambda(X,\mu) = \mu$ for every $(X,\mu) \in {}_{\mathbb{Q}}\mathscr{C}$.

A monad $\mathbb{Q}$ on $\mathscr{C}$ gives rise to an adjunction $(F,U) := ({}_{\mathbb{Q}}F, {}_{\mathbb{Q}}U)$ where $U : {}_{\mathbb{Q}}\mathscr{C} \to \mathscr{C}$ is the *forgetful functor* and $F : \mathscr{C} \to {}_{\mathbb{Q}}\mathscr{C}$ is the *free functor*. Explicitly:

$$U(X,\mu) := X, \quad Uf := f \qquad \text{and} \qquad FX := (QX, mX), \quad Ff := Qf.$$

Note that

- $UF = Q$.
- The unit of the adjunction (F,U) is given by the unit $u : \mathrm{Id}_{\mathscr{C}} \to UF = Q$ of the monad $\mathbb{Q}$.
- The counit $\lambda : FU \to \mathrm{Id}_{{}_{\mathbb{Q}}\mathscr{C}}$ is uniquely determined by the equality $U\lambda(X,\mu) = \mu$ for every $(X,\mu) \in {}_{\mathbb{Q}}\mathscr{C}$.
- the forgetful functor $U : {}_{\mathbb{Q}}\mathscr{C} \to \mathscr{C}$ is faithful and reflects isomorphisms.

Monadic decomposition

Let $(L : \mathscr{B} \to \mathscr{A}, R : \mathscr{A} \to \mathscr{B})$ be an adjunction with unit η and counit ε.

Monadic decomposition

Let $(L : \mathscr{B} \to \mathscr{A}, R : \mathscr{A} \to \mathscr{B})$ be an adjunction with unit η and counit ε. Then $(RL, R\varepsilon L, \eta)$ is a monad on $\mathscr{B}$ and we can assign to it the so-called comparison functor $K : \mathscr{A} \to {}_{RL}\mathscr{B}$ which is defined by

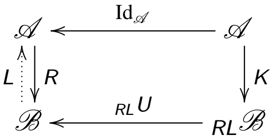
$$KX := (RX, R\varepsilon X) \qquad \text{and} \qquad Kf := Rf.$$

Monadic decomposition

Let $(L : \mathscr{B} \to \mathscr{A}, R : \mathscr{A} \to \mathscr{B})$ be an adjunction with unit η and counit ε. Then $(RL, R\varepsilon L, \eta)$ is a monad on $\mathscr{B}$ and we can assign to it the so-called comparison functor $K : \mathscr{A} \to {}_{RL}\mathscr{B}$ which is defined by

$$KX := (RX, R\varepsilon X) \qquad \text{and} \qquad Kf := Rf.$$

We have this commutative diagram.

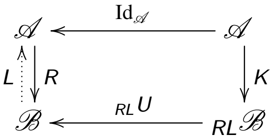


Monadic decomposition

Let $(L : \mathscr{B} \to \mathscr{A}, R : \mathscr{A} \to \mathscr{B})$ be an adjunction with unit η and counit ε. Then $(RL, R\varepsilon L, \eta)$ is a monad on $\mathscr{B}$ and we can assign to it the so-called comparison functor $K : \mathscr{A} \to {}_{RL}\mathscr{B}$ which is defined by

$$KX := (RX, R\varepsilon X) \qquad \text{and} \qquad Kf := Rf.$$

We have this commutative diagram.



Definition

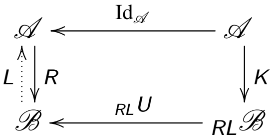
A functor R is **monadic** (tripleable in Beck's terminology) if it has a left adjoint L such that the functor K, as above, is an equivalence of categories.

Monadic decomposition

Let $(L : \mathscr{B} \to \mathscr{A}, R : \mathscr{A} \to \mathscr{B})$ be an adjunction with unit η and counit ε. Then $(RL, R\varepsilon L, \eta)$ is a monad on $\mathscr{B}$ and we can assign to it the so-called comparison functor $K : \mathscr{A} \to {}_{RL}\mathscr{B}$ which is defined by

$$KX := (RX, R\varepsilon X) \qquad \text{and} \qquad Kf := Rf.$$

We have this commutative diagram.



Definition

A functor R is **monadic** (tripleable in Beck's terminology) if it has a left adjoint L such that the functor K, as above, is an equivalence of categories.

An easy example

The functor ${}_{RL}U$ is always monadic!!!

Let us consider again our diagram but changing the notation as follows

Let us consider again our diagram but changing the notation as follows

Old notation

$$
\begin{array}{ccc}
\mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\
{\scriptstyle L}\uparrow\downarrow{\scriptstyle R} & & \downarrow{\scriptstyle K} \\
\mathscr{B} & \xleftarrow{{}_{RL}U} & {}_{RL}\mathscr{B}
\end{array}
$$

New notation

$$
\begin{array}{ccc}
\mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\
{\scriptstyle L_0}\uparrow\downarrow{\scriptstyle R_0} & & \downarrow{\scriptstyle R_1} \\
\mathscr{B}_0 & \xleftarrow{U_{0,1}} & \mathscr{B}_1
\end{array}
$$

Let us consider again our diagram but changing the notation as follows

Old notation

$$\begin{array}{ccc} \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\ L \uparrow \downarrow R & & \downarrow K \\ \mathscr{B} & \xleftarrow{{}_{RL}U} & {}_{RL}\mathscr{B} \end{array}$$

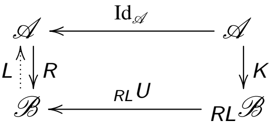
New notation

$$\begin{array}{ccc} \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\ L_0 \uparrow \downarrow R_0 & & \downarrow R_1 \\ \mathscr{B}_0 & \xleftarrow{U_{0,1}} & \mathscr{B}_1 \end{array}$$

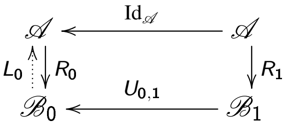
Suppose that R_1 has a left adjoint L_1.

Let us consider again our diagram but changing the notation as follows

Old notation

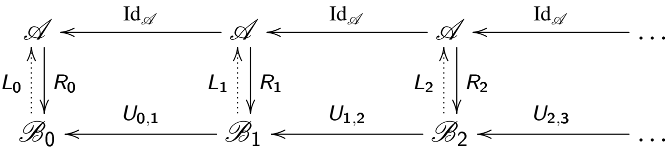


New notation



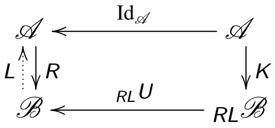
Suppose that R_1 has a left adjoint L_1.

Then we can consider (L_1, R_1) as a starting adjunction and, if we are lucky, extend the diagram as follows

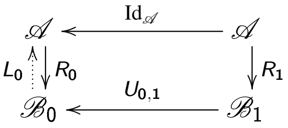


Let us consider again our diagram but changing the notation as follows

Old notation

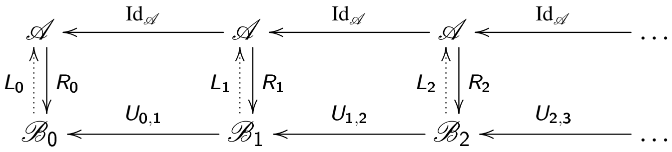


New notation



Suppose that R_1 has a left adjoint L_1.

Then we can consider (L_1, R_1) as a starting adjunction and, if we are lucky, extend the diagram as follows



For all $i \in \mathbb{N}$, the unit and counit of the adjunction (L_i, R_i) will be denoted by η_i and ε_i respectively.

We are very lucky if $\exists$ minimal $N \in \mathbb{N}$ such that L_N is full and faithful.

We are very lucky if $\exists$ minimal $N \in \mathbb{N}$ such that L_N is full and faithful.

$$\begin{array}{ccccc}
\mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\
L_0 \uparrow \downarrow R_0 & & L_1 \uparrow \downarrow R_1 & & L_2 \uparrow \downarrow R_2 \\
\mathscr{B}_0 & \xleftarrow[U_{0,1}]{} & \mathscr{B}_1 & \xleftarrow[U_{1,2}]{} & \mathscr{B}_2
\end{array}
\qquad
\begin{array}{ccccc}
\cdots & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\
 & & L_N \uparrow \downarrow R_N & & \downarrow R_{N+1} \\
\cdots & \xleftarrow[U_{N-1,N}]{} & \mathscr{B}_N & \xleftarrow[U_{N,N+1}]{} & \mathscr{B}_{N+1}
\end{array}$$

We are very lucky if $\exists$ minimal $N \in \mathbb{N}$ such that L_N is full and faithful.

$$
\begin{array}{ccccc}
\mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\
L_0 \uparrow \downarrow R_0 & & L_1 \uparrow \downarrow R_1 & & L_2 \uparrow \downarrow R_2 \\
\mathscr{B}_0 & \xleftarrow[U_{0,1}]{} & \mathscr{B}_1 & \xleftarrow[U_{1,2}]{} & \mathscr{B}_2
\end{array}
\qquad
\begin{array}{ccccc}
\cdots & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\
 & & L_N \uparrow \downarrow R_N & & \downarrow R_{N+1} \\
\cdots & \xleftarrow[U_{N-1,N}]{} & \mathscr{B}_N & \xleftarrow[U_{N,N+1}]{} & \mathscr{B}_{N+1}
\end{array}
$$

Lemma

L_N f.f. $\Leftrightarrow U_{N,N+1} : \mathscr{B}_{N+1} \to \mathscr{B}_N$ is an isomorphism of categories.

We are very lucky if $\exists$ minimal $N \in \mathbb{N}$ such that L_N is full and faithful.

$$
\begin{array}{ccccc}
\mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\
{\scriptstyle L_0} \uparrow \downarrow {\scriptstyle R_0} & & {\scriptstyle L_1} \uparrow \downarrow {\scriptstyle R_1} & & {\scriptstyle L_2} \uparrow \downarrow {\scriptstyle R_2} \\
\mathscr{B}_0 & \xleftarrow[U_{0,1}]{} & \mathscr{B}_1 & \xleftarrow[U_{1,2}]{} & \mathscr{B}_2
\end{array}
\qquad
\begin{array}{ccccc}
\cdots & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\
 & & {\scriptstyle L_N} \uparrow \downarrow {\scriptstyle R_N} & & \downarrow {\scriptstyle R_{N+1}} \\
\cdots & \xleftarrow[U_{N-1,N}]{} & \mathscr{B}_N & \xleftarrow[U_{N,N+1}]{} & \mathscr{B}_{N+1}
\end{array}
$$

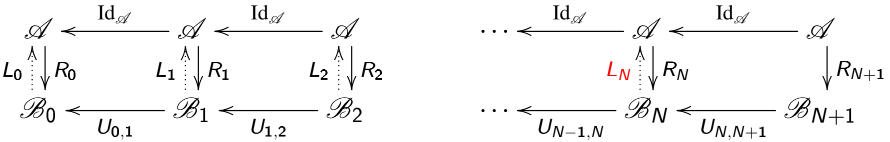
Lemma

L_N f.f. $\Leftrightarrow U_{N,N+1} : \mathscr{B}_{N+1} \to \mathscr{B}_N$ is an isomorphism of categories.

Proof.

It relies on the fact that, by Rafael Theorem, L_N f.f. $\Leftrightarrow$ the unit η_N is an isomorphism. □

We are very lucky if $\exists$ minimal $N \in \mathbb{N}$ such that L_N is full and faithful.



Lemma

L_N f.f. $\Leftrightarrow U_{N,N+1} : \mathscr{B}_{N+1} \to \mathscr{B}_N$ is an isomorphism of categories.

Proof.

It relies on the fact that, by Rafael Theorem, L_N f.f. $\Leftrightarrow$ the unit η_N is an isomorphism. □

Thus, if such an N exists, then $\mathscr{B}_{N+1} \cong \mathscr{B}_N$ and the diagram is stationary.

Note that, by the commutativity of the diagram,

$$
\begin{array}{ccccccccc}
\mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} & & \cdots & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\
L_0 \uparrow \downarrow R_0 & & L_1 \uparrow \downarrow R_1 & & L_2 \uparrow \downarrow R_2 & & & & L_N \uparrow \downarrow R_N \\
\mathscr{B}_0 & \xleftarrow{U_{0,1}} & \mathscr{B}_1 & \xleftarrow{U_{1,2}} & \mathscr{B}_2 & & \cdots & \xleftarrow{U_{N-1,N}} & \mathscr{B}_N
\end{array}
$$

Note that, by the commutativity of the diagram,

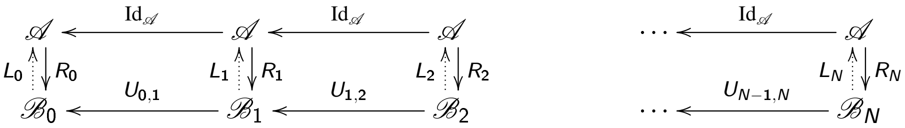
$$\begin{array}{ccccccccc}
\mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} & & \cdots & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\
{\scriptstyle L_0}\uparrow\downarrow{\scriptstyle R_0} & & {\scriptstyle L_1}\uparrow\downarrow{\scriptstyle R_1} & & {\scriptstyle L_2}\uparrow\downarrow{\scriptstyle R_2} & & & & {\scriptstyle L_N}\uparrow\downarrow{\scriptstyle R_N} \\
\mathscr{B}_0 & \xleftarrow{U_{0,1}} & \mathscr{B}_1 & \xleftarrow{U_{1,2}} & \mathscr{B}_2 & & \cdots & \xleftarrow{U_{N-1,N}} & \mathscr{B}_N
\end{array}$$

we can write

$$R = R_0 = U_{0,1} \circ U_{1,2} \cdots U_{N-1,N} \circ R_N$$

where $U_{0,1}, U_{1,2}, \cdots, U_{N-1,N}$ are N monadic functors but not category isomorphisms.

Note that, by the commutativity of the diagram,



we can write

$$R = R_0 = U_{0,1} \circ U_{1,2} \cdots U_{N-1,N} \circ R_N$$

where $U_{0,1}, U_{1,2}, \cdots, U_{N-1,N}$ are N monadic functors but not category isomorphisms.

Moreover this is a maximal decomposition of this form.

Note that, by the commutativity of the diagram,

$$\begin{array}{ccccccccc}
\mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} & & \cdots & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\
L_0 \uparrow \downarrow R_0 & & L_1 \uparrow \downarrow R_1 & & L_2 \uparrow \downarrow R_2 & & & & L_N \uparrow \downarrow R_N \\
\mathscr{B}_0 & \xleftarrow{U_{0,1}} & \mathscr{B}_1 & \xleftarrow{U_{1,2}} & \mathscr{B}_2 & & \cdots & \xleftarrow{U_{N-1,N}} & \mathscr{B}_N
\end{array}$$

we can write

$$R = R_0 = U_{0,1} \circ U_{1,2} \cdots U_{N-1,N} \circ R_N$$

where $U_{0,1}, U_{1,2}, \cdots, U_{N-1,N}$ are N monadic functors but not category isomorphisms.

Moreover this is a maximal decomposition of this form.

For this reason we will say that such an R has a

monadic decomposition of (monadic) length N.

The investigation of monadic decompositions goes back to

[MS] J. L. MacDonald, A. Stone, *The tower and regular decomposition*. Cahiers Topologie Géom. Différentielle **23**. (1982), no. 2, 197-213.

[AHW] J. Adámek, H. Herrlich, W. Tholen, *Monadic decompositions*. J. Pure Appl. Algebra **59** (1989), no. 2, 111-123.

The investigation of monadic decompositions goes back to

[MS] J. L. MacDonald, A. Stone, *The tower and regular decomposition*. Cahiers Topologie Géom. Différentielle **23**. (1982), no. 2, 197-213.

[AHW] J. Adámek, H. Herrlich, W. Tholen, *Monadic decompositions*. J. Pure Appl. Algebra **59** (1989), no. 2, 111-123.

Note that the notion of comonadic decomposition of (comonadic) length N can be easily introduced and to distinguish it we will use the notations

$$(L^n, R^n)$$

with superscripts and require that R^N be full and faithful.

The investigation of monadic decompositions goes back to

[MS] J. L. MacDonald, A. Stone, *The tower and regular decomposition*. Cahiers Topologie Géom. Différentielle **23**. (1982), no. 2, 197-213.

[AHW] J. Adámek, H. Herrlich, W. Tholen, *Monadic decompositions*. J. Pure Appl. Algebra **59** (1989), no. 2, 111-123.

Note that the notion of comonadic decomposition of (comonadic) length N can be easily introduced and to distinguish it we will use the notations

$$(L^n, R^n)$$

with superscripts and require that R^N be full and faithful.

Next aim is to investigate some properties of functors with a finite length monadic decomposition.

The image of a functor

Notation

Given a functor $R:\mathscr{A}\rightarrow\mathscr{B}$, denote by $\mathrm{Im}R$ *the full subcategory of* $\mathscr{B}$ *consisting of objects* $B\in\mathscr{B}$ *such that* $B\cong RA$ *for some object* $A\in\mathscr{A}$.

The image of a functor

Notation

Given a functor $R:\mathscr{A}\to\mathscr{B}$, denote by $\mathrm{Im}R$ the full subcategory of $\mathscr{B}$ consisting of objects $B\in\mathscr{B}$ such that $B\cong RA$ for some object $A\in\mathscr{A}$.

Recall that a functor $R:\mathscr{A}\to\mathscr{B}$ is essentially surjective if $\mathrm{Im}R=\mathscr{B}$.

The image of a functor

Notation

Given a functor $R : \mathscr{A} \to \mathscr{B}$, denote by $\mathrm{Im}R$ *the full subcategory of* $\mathscr{B}$ *consisting of objects $B \in \mathscr{B}$ such that $B \cong RA$ for some object $A \in \mathscr{A}$.*

Recall that a functor $R : \mathscr{A} \to \mathscr{B}$ is essentially surjective if $\mathrm{Im}R = \mathscr{B}$.

Proposition

Given a functor $R : \mathscr{A} \to \mathscr{B}$ having a monadic decomposition of length $N \in \mathbb{N}$, then $\mathrm{Im}R = \mathrm{Im}U_{0,N}$, where we set $U_{0,N} := U_{0,1} \circ U_{1,2} \circ \cdots \circ U_{N-1,N}$.

The image of a functor

Notation

Given a functor $R : \mathscr{A} \to \mathscr{B}$, denote by $\mathrm{Im}R$ the full subcategory of $\mathscr{B}$ consisting of objects $B \in \mathscr{B}$ such that $B \cong RA$ for some object $A \in \mathscr{A}$.

Recall that a functor $R : \mathscr{A} \to \mathscr{B}$ is essentially surjective if $\mathrm{Im}R = \mathscr{B}$.

Proposition

Given a functor $R : \mathscr{A} \to \mathscr{B}$ having a monadic decomposition of length $N \in \mathbb{N}$, then $\mathrm{Im}R = \mathrm{Im}U_{0,N}$, where we set $U_{0,N} := U_{0,1} \circ U_{1,2} \circ \cdots \circ U_{N-1,N}$.

Proof.

The monadic decomposition rewrites as $R = U_{0,N} \circ R_N$.

The image of a functor

Notation

Given a functor $R : \mathscr{A} \to \mathscr{B}$, denote by $\mathrm{Im}R$ the full subcategory of $\mathscr{B}$ consisting of objects $B \in \mathscr{B}$ such that $B \cong RA$ for some object $A \in \mathscr{A}$.

Recall that a functor $R : \mathscr{A} \to \mathscr{B}$ is essentially surjective if $\mathrm{Im}R = \mathscr{B}$.

Proposition

Given a functor $R : \mathscr{A} \to \mathscr{B}$ having a monadic decomposition of length $N \in \mathbb{N}$, then $\mathrm{Im}R = \mathrm{Im}U_{0,N}$, where we set $U_{0,N} := U_{0,1} \circ U_{1,2} \circ \cdots \circ U_{N-1,N}$.

Proof.

The monadic decomposition rewrites as $R = U_{0,N} \circ R_N$. By assumption the left adjoint L_N of R_N is full and faithful so that the unit $\eta_N : \mathrm{Id}_{\mathscr{B}_N} \to R_N L_N$ is an isomorphism.

The image of a functor

Notation

Given a functor $R : \mathscr{A} \to \mathscr{B}$*, denote by* $\mathrm{Im}R$ *the full subcategory of* $\mathscr{B}$ *consisting of objects* $B \in \mathscr{B}$ *such that* $B \cong RA$ *for some object* $A \in \mathscr{A}$*.*

Recall that a functor $R : \mathscr{A} \to \mathscr{B}$ is essentially surjective if $\mathrm{Im}R = \mathscr{B}$.

Proposition

Given a functor $R : \mathscr{A} \to \mathscr{B}$ *having a monadic decomposition of length* $N \in \mathbb{N}$*, then* $\mathrm{Im}R = \mathrm{Im}U_{0,N}$*, where we set* $U_{0,N} := U_{0,1} \circ U_{1,2} \circ \cdots \circ U_{N-1,N}$*.*

Proof.

The monadic decomposition rewrites as $R = U_{0,N} \circ R_N$. By assumption the left adjoint L_N of R_N is full and faithful so that the unit $\eta_N : \mathrm{Id}_{\mathscr{B}_N} \to R_N L_N$ is an isomorphism. Hence R_N is essentially surjective and we get $\mathrm{Im}R = \mathrm{Im}U_{0,N}$. □

Hence, monadic decomposition is a good tool to determine images of functors.

Hence, monadic decomposition is a good tool to determine images of functors. Let us give an example connected to Descent theory.

Hence, monadic decomposition is a good tool to determine images of functors. Let us give an example connected to Descent theory.

Let A, B be rings. Given a (B, A)-bimodule M, consider the adjunction

$$L : \mathscr{M}_B \to \mathscr{M}_A : X \mapsto X \otimes_B M \qquad R : \mathscr{M}_A \to \mathscr{M}_B : Y \mapsto \mathrm{Hom}_A(M, Y).$$

Here $\mathscr{M}_E = \mathrm{Mod}\text{-}E$ category of right modules over the ring $E = A, B$.

Hence, monadic decomposition is a good tool to determine images of functors. Let us give an example connected to Descent theory.

Let A, B be rings. Given a (B, A)-bimodule M, consider the adjunction

$$L : \mathscr{M}_B \to \mathscr{M}_A : X \mapsto X \otimes_B M \qquad R : \mathscr{M}_A \to \mathscr{M}_B : Y \mapsto \mathrm{Hom}_A(M, Y).$$

Here $\mathscr{M}_E = \mathrm{Mod}\text{-}E$ category of right modules over the ring $E = A, B$.

Now the comparison functors R_1 and L^1 have adjoints as follows.

$$\begin{array}{ccccc}
(\mathscr{M}_A)^1 & \overset{U^{0,1}}{\dashrightarrow} & \mathscr{M}_A & \overset{\mathrm{Id}_{\mathscr{M}_A}}{\longleftarrow} & \mathscr{M}_A \\
L^1 \uparrow\downarrow R^1 & & L \uparrow\downarrow R & & L_1 \uparrow\downarrow R_1 \\
\mathscr{M}_B & \underset{\mathrm{Id}_{\mathscr{M}_B}}{\dashrightarrow} & \mathscr{M}_B & \overset{U_{0,1}}{\longleftarrow} & (\mathscr{M}_B)_1
\end{array}$$

Hence, monadic decomposition is a good tool to determine images of functors. Let us give an example connected to Descent theory.

Let A, B be rings. Given a (B, A)-bimodule M, consider the adjunction

$$L : \mathcal{M}_B \to \mathcal{M}_A : X \mapsto X \otimes_B M \qquad R : \mathcal{M}_A \to \mathcal{M}_B : Y \mapsto \mathrm{Hom}_A(M, Y).$$

Here $\mathcal{M}_E = \mathrm{Mod}\text{-}E$ category of right modules over the ring $E = A, B$.

Now the comparison functors R_1 and L^1 have adjoints as follows.

$$\begin{array}{ccccc}
(\mathcal{M}_A)^1 & \overset{U^{0,1}}{\dashrightarrow} & \mathcal{M}_A & \overset{\mathrm{Id}_{\mathcal{M}_A}}{\longleftarrow} & \mathcal{M}_A \\
L^1 \uparrow \downarrow R^1 & & L \uparrow \downarrow R & & L_1 \uparrow \downarrow R_1 \\
\mathcal{M}_B & \underset{\mathrm{Id}_{\mathcal{M}_B}}{\dashrightarrow} & \mathcal{M}_B & \overset{U_{0,1}}{\longleftarrow} & (\mathcal{M}_B)_1
\end{array}$$

Focus on the right-hand side diagram and assume that M_A is projective.

Hence, monadic decomposition is a good tool to determine images of functors. Let us give an example connected to Descent theory.

Let A, B be rings. Given a (B, A)-bimodule M, consider the adjunction

$$L : \mathcal{M}_B \to \mathcal{M}_A : X \mapsto X \otimes_B M \qquad R : \mathcal{M}_A \to \mathcal{M}_B : Y \mapsto \mathrm{Hom}_A(M, Y).$$

Here $\mathcal{M}_E = \mathrm{Mod}\text{-}E$ category of right modules over the ring $E = A, B$.

Now the comparison functors R_1 and L^1 have adjoints as follows.

$$\begin{array}{ccccc}
(\mathcal{M}_A)^1 & \xrightarrow{U^{0,1}} & \mathcal{M}_A & \xleftarrow{\mathrm{Id}_{\mathcal{M}_A}} & \mathcal{M}_A \\
L^1 \uparrow \downarrow R^1 & & L \uparrow \downarrow R & & L_1 \uparrow \downarrow R_1 \\
\mathcal{M}_B & \xrightarrow[\mathrm{Id}_{\mathcal{M}_B}]{} & \mathcal{M}_B & \xleftarrow{U_{0,1}} & (\mathcal{M}_B)_1
\end{array}$$

Focus on the right-hand side diagram and assume that M_A is projective. Then $R = R_0$ is exact so that, Beck's Theorem ensures that L_1 is full and faithful, and R has a monadic decomposition of length at most 1.

$$\begin{array}{ccccc}
(\mathscr{M}_A)^1 & \overset{U^{0,1}}{\dashrightarrow} & \mathscr{M}_A & \overset{\mathrm{Id}_{\mathscr{M}_A}}{\longleftarrow} & \mathscr{M}_A \\
L^1 \uparrow \downarrow R^1 & & L \uparrow \downarrow R & & L_1 \uparrow \downarrow R_1 \\
\mathscr{M}_B & \underset{\mathrm{Id}_{\mathscr{M}_B}}{\dashrightarrow} & \mathscr{M}_B & \overset{U_{0,1}}{\longleftarrow} & (\mathscr{M}_B)_1
\end{array}$$

$$\begin{array}{ccccc}
(\mathscr{M}_A)^1 & \xrightarrow{U^{0,1}} & \mathscr{M}_A & \xleftarrow{\mathrm{Id}_{\mathscr{M}_A}} & \mathscr{M}_A \\
{\scriptstyle L^1}\uparrow\downarrow{\scriptstyle R^1} & & {\scriptstyle L}\uparrow\downarrow{\scriptstyle R} & & {\scriptstyle L_1}\uparrow\downarrow{\scriptstyle R_1} \\
\mathscr{M}_B & \xrightarrow[\mathrm{Id}_{\mathscr{M}_B}]{} & \mathscr{M}_B & \xleftarrow{U_{0,1}} & (\mathscr{M}_B)_1
\end{array}$$

Focus on the left-hand side diagram and assume $_BM$ is flat.

$$_BM \text{ flat} \Rightarrow L = L^0 \text{ exact} \overset{\text{dual Beck's Thm.}}{\Rightarrow} R^1 \text{ full and faithful.}$$

$$
\begin{array}{ccccc}
(\mathscr{M}_A)^1 & \overset{U^{0,1}}{\dashrightarrow} & \mathscr{M}_A & \overset{\mathrm{Id}_{\mathscr{M}_A}}{\longleftarrow} & \mathscr{M}_A \\
L^1 \Uparrow \downarrow R^1 & & L \Uparrow \downarrow R & & L_1 \Uparrow \downarrow R_1 \\
\mathscr{M}_B & \underset{\mathrm{Id}_{\mathscr{M}_B}}{\dashrightarrow} & \mathscr{M}_B & \overset{U_{0,1}}{\longleftarrow} & (\mathscr{M}_B)_1
\end{array}
$$

Focus on the left-hand side diagram and assume ${}_BM$ is flat.

$$
{}_BM \text{ flat} \Rightarrow L = L^0 \text{ exact} \overset{\text{dual Beck's Thm.}}{\Rightarrow} R^1 \text{ full and faithful.}
$$

Therefore, L admits a comonadic decomposition of length at most 1.

$$
\begin{array}{ccccc}
(\mathscr{M}_A)^1 & \overset{U^{0,1}}{\dashrightarrow} & \mathscr{M}_A & \overset{\mathrm{Id}_{\mathscr{M}_A}}{\longleftarrow} & \mathscr{M}_A \\
L^1 \uparrow \downarrow R^1 & & L \uparrow \downarrow R & & L_1 \uparrow \downarrow R_1 \\
\mathscr{M}_B & \underset{\mathrm{Id}_{\mathscr{M}_B}}{\dashrightarrow} & \mathscr{M}_B & \overset{U_{0,1}}{\longleftarrow} & (\mathscr{M}_B)_1
\end{array}
$$

Focus on the left-hand side diagram and assume ${}_BM$ is flat.

$$ {}_BM \text{ flat} \Rightarrow L = L^0 \text{ exact} \overset{\text{dual Beck's Thm.}}{\Rightarrow} R^1 \text{ full and faithful.} $$

Therefore, L admits a comonadic decomposition of length at most 1.

CONSEQUENCE: $\mathrm{Im} L = \mathrm{Im} U^{0,1}$ i.e. the objects of $\mathscr{M}_A$ which are isomorphic to objects of the form $LX = X \otimes_B M$, for some $X \in \mathscr{M}_B$, are exactly those of the form $U^{0,1}X^1$ where $X^1 \in (\mathscr{M}_A)^1$.

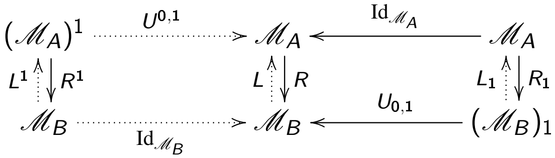
$$
\begin{array}{ccccc}
(\mathscr{M}_A)^1 & \overset{U^{0,1}}{\dashrightarrow} & \mathscr{M}_A & \overset{\mathrm{Id}_{\mathscr{M}_A}}{\longleftarrow} & \mathscr{M}_A \\
L^1 \uparrow \downarrow R^1 & & L \uparrow \downarrow R & & L_1 \uparrow \downarrow R_1 \\
\mathscr{M}_B & \underset{\mathrm{Id}_{\mathscr{M}_B}}{\dashrightarrow} & \mathscr{M}_B & \overset{U_{0,1}}{\longleftarrow} & (\mathscr{M}_B)_1
\end{array}
$$

Focus on the left-hand side diagram and assume ${}_BM$ is flat.

$$ {}_BM \text{ flat} \Rightarrow L = L^0 \text{ exact} \overset{\text{dual Beck's Thm.}}{\Rightarrow} R^1 \text{ full and faithful.} $$

Therefore, L admits a comonadic decomposition of length at most 1.

CONSEQUENCE: $\mathrm{Im}L = \mathrm{Im}U^{0,1}$ i.e. the objects of $\mathscr{M}_A$ which are isomorphic to objects of the form $LX = X \otimes_B M$, for some $X \in \mathscr{M}_B$, are exactly those of the form $U^{0,1}X^1$ where $X^1 \in (\mathscr{M}_A)^1$. Hence the category $(\mathscr{M}_A)^1$ solves the descent problem for modules.



Focus on the left-hand side diagram and assume ${}_B M$ is flat.

$$ {}_B M \text{ flat} \Rightarrow L = L^0 \text{ exact} \overset{\text{dual Beck's Thm.}}{\Rightarrow} R^1 \text{ full and faithful.} $$

Therefore, L admits a comonadic decomposition of length at most 1.

CONSEQUENCE: $\mathrm{Im} L = \mathrm{Im} U^{0,1}$ i.e. the objects of $\mathscr{M}_A$ which are isomorphic to objects of the form $LX = X \otimes_B M$, for some $X \in \mathscr{M}_B$, are exactly those of the form $U^{0,1}X^1$ where $X^1 \in (\mathscr{M}_A)^1$. Hence the category $(\mathscr{M}_A)^1$ solves the descent problem for modules.

Note: when M_A is also finitely generated and projective, then $(\mathscr{M}_A)^1$ is precisely the category of comodules over the A-coring $M^* \otimes_B M$ (the so-called comatrix coring associated to ${}_B M_A$).

Idempotent monads and adjunctions

The notion of an idempotent monad is, as we will see below, related with the monadic length of a functor.

Idempotent monads and adjunctions

The notion of an idempotent monad is, as we will see below, related with the monadic length of a functor.

Definition

- A monad (Q, m, u) is idempotent if m is an isomorphism.

Idempotent monads and adjunctions

The notion of an idempotent monad is, as we will see below, related with the monadic length of a functor.

Definition

- A monad (Q, m, u) is idempotent if m is an isomorphism.
- An adjunction (L, R) is idempotent if the associated monad is.

Idempotent monads and adjunctions

The notion of an idempotent monad is, as we will see below, related with the monadic length of a functor.

Definition

- A monad (Q, m, u) is idempotent if m is an isomorphism.
- An adjunction (L, R) is idempotent if the associated monad is.

There are several basic characterizations of idempotent adjunctions, see

[AT] H. Appelgate, M. Tierney, *Categories with models*. 1969 Sem. on Triples and Categorical Homology Theory (ETH, zürich, 1966/67) pp. 156–244 Springer, Berlin.

[MS] J. L. MacDonald, A. Stone, *The tower and regular decomposition*. Cahiers Topologie Géom. Différentielle **23** (1982), no. 2, 197-213.

Idempotent monads and adjunctions

The notion of an idempotent monad is, as we will see below, related with the monadic length of a functor.

Definition

- A monad (Q, m, u) is idempotent if m is an isomorphism.
- An adjunction (L, R) is idempotent if the associated monad is.

There are several basic characterizations of idempotent adjunctions, see

[AT] H. Appelgate, M. Tierney, *Categories with models*. 1969 Sem. on Triples and Categorical Homology Theory (ETH, zürich, 1966/67) pp. 156–244 Springer, Berlin.

[MS] J. L. MacDonald, A. Stone, *The tower and regular decomposition*. Cahiers Topologie Géom. Différentielle **23** (1982), no. 2, 197-213.

In particular, idempotency of an adjunction means equivalently that any one of $\varepsilon L, R\varepsilon, \eta R, L\eta$ is an isomorphism ([MS, Proposition 2.8]).

Proposition

For our diagram,
in the general case,
TFAE.

$$\begin{array}{ccc} \mathscr{A} & \xleftarrow{\ \mathrm{Id}_{\mathscr{A}}\ } & \mathscr{A} \\ L \Big\uparrow \Big\downarrow R & & \Big\downarrow R_1 \\ \mathscr{B} & \xleftarrow{\ U_{0,1}\ } & \mathscr{B}_1 \end{array}$$

Proposition

For our diagram,
in the general case,
TFAE.

$$\begin{array}{ccc} \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\ {\scriptstyle L}\uparrow\downarrow{\scriptstyle R} & & \downarrow{\scriptstyle R_1} \\ \mathscr{B} & \xleftarrow{U_{0,1}} & \mathscr{B}_1 \end{array}$$

(a) (L, R) *is idempotent.*

Proposition

For our diagram,
in the general case,
TFAE.

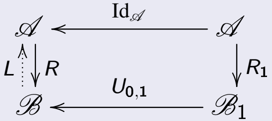
$$\begin{array}{ccc} \mathscr{A} & \xleftarrow{\mathrm{Id}_{\mathscr{A}}} & \mathscr{A} \\ L \uparrow \downarrow R & & \downarrow R_1 \\ \mathscr{B} & \xleftarrow{U_{0,1}} & \mathscr{B}_1 \end{array}$$

(a) (L,R) *is idempotent.*

(b) $\forall (X, RLX \xrightarrow{\mu} X) \in \mathscr{B}_1$ *we have that* μ *is an isomorphism.*

Proposition

For our diagram,
in the general case,
TFAE.



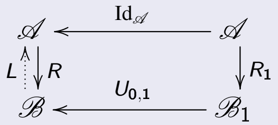
(a) (L, R) *is idempotent.*

(b) $\forall (X, RLX \xrightarrow{\mu} X) \in \mathscr{B}_1$ *we have that* μ *is an isomorphism.*

(c) $L_1 := LU_{0,1}$ *is a left adjoint of* R_1, $\eta U_{0,1} = U_{0,1}\eta_1$ *and* $\varepsilon_1 = \varepsilon$.

Proposition

For our diagram,
in the general case,
TFAE.



(a) (L,R) *is idempotent.*

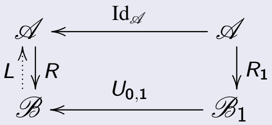
(b) $\forall (X, RLX \xrightarrow{\mu} X) \in \mathscr{B}_1$ *we have that* μ *is an isomorphism.*

(c) $L_1 := LU_{0,1}$ *is a left adjoint of* R_1*,* $\eta U_{0,1} = U_{0,1}\eta_1$ *and* $\varepsilon_1 = \varepsilon$*.*

Moreover, if one of these conditions holds, then L_1 *is full and faithful.*

Proposition

For our diagram, in the general case, TFAE.



(a) (L, R) *is idempotent.*

(b) $\forall (X, RLX \xrightarrow{\mu} X) \in \mathscr{B}_1$ *we have that* μ *is an isomorphism.*

(c) $L_1 := LU_{0,1}$ *is a left adjoint of* R_1, $\eta U_{0,1} = U_{0,1}\eta_1$ *and* $\varepsilon_1 = \varepsilon$.

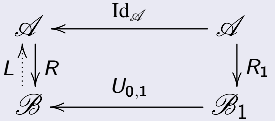
Moreover, if one of these conditions holds, then L_1 *is full and faithful.*

Corollary

- (L, R) *idempotent* $\Longrightarrow R$ *has a monadic decomposition of length* ≤ 1.

Proposition

For our diagram, in the general case, TFAE.



(a) (L,R) *is idempotent.*

(b) $\forall (X, RLX \xrightarrow{\mu} X) \in \mathscr{B}_1$ *we have that* μ *is an isomorphism.*

(c) $L_1 := LU_{0,1}$ *is a left adjoint of* R_1, $\eta U_{0,1} = U_{0,1}\eta_1$ *and* $\varepsilon_1 = \varepsilon$.

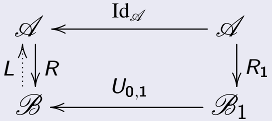
Moreover, if one of these conditions holds, then L_1 *is full and faithful.*

Corollary

- (L,R) *idempotent* $\Longrightarrow$ R *has a monadic decomposition of length* ≤ 1.
- (L_i, R_i) *idempotent for some* $i \in \mathbb{N} \Longrightarrow$ R *has a monadic decomposition of length* $\leq i+1$.

Proposition

For our diagram, in the general case, TFAE.



(a) (L,R) *is idempotent.*

(b) $\forall (X, RLX \xrightarrow{\mu} X) \in \mathscr{B}_1$ *we have that* μ *is an isomorphism.*

(c) $L_1 := LU_{0,1}$ *is a left adjoint of* R_1, $\eta U_{0,1} = U_{0,1}\eta_1$ *and* $\varepsilon_1 = \varepsilon$.

Moreover, if one of these conditions holds, then L_1 *is full and faithful.*

Corollary

- (L,R) *idempotent* $\Longrightarrow$ R *has a monadic decomposition of length* ≤ 1.
- (L_i, R_i) *idempotent for some* $i \in \mathbb{N} \Longrightarrow R$ *has a monadic decomposition of length* $\leq i+1$.

Moreover one checks that (L,R) idempotent $\Longleftrightarrow \eta U_{0,1}$ is an isomorphism.

Vector spaces and bialgebras

Fix an arbitrary field $\mathbb{k}$.

Vector spaces and bialgebras

Fix an arbitrary field $\Bbbk$. All vector spaces, (co)algebras and bialgebras will be over $\Bbbk$.

Vector spaces and bialgebras

Fix an arbitrary field $\Bbbk$. All vector spaces, (co)algebras and bialgebras will be over $\Bbbk$.

Let $\mathbf{Vec}$:= category of vector spaces and $\mathbf{Bialg}$:= category of bialgebras.

Vector spaces and bialgebras

Fix an arbitrary field $\mathbb{k}$. All vector spaces, (co)algebras and bialgebras will be over $\mathbb{k}$.

Let **Vec** := category of vector spaces and **Bialg** := category of bialgebras.

Consider the following well-known adjunction

$$\begin{array}{ccc} \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\ {\scriptstyle T}\uparrow\downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P_1} \\ \mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 \end{array}$$

Vector spaces and bialgebras

Fix an arbitrary field $\Bbbk$. All vector spaces, (co)algebras and bialgebras will be over $\Bbbk$.

Let **Vec** := category of vector spaces and **Bialg** := category of bialgebras.

Consider the following well-known adjunction

$$\begin{array}{ccc} \mathbf{Bialg} & \xleftarrow{\ \mathrm{Id}_{\mathbf{Bialg}}\ } & \mathbf{Bialg} \\ {\scriptstyle T}\uparrow\downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P_1} \\ \mathbf{Vec} & \xleftarrow{\ U_{0,1}\ } & \mathbf{Vec}_1 \end{array}$$

Here P sends a bialgebra B to its subspace of primitive elements

$$PB := \{b \in B \mid \Delta(b) = 1 \otimes b + b \otimes 1\},$$

Vector spaces and bialgebras

Fix an arbitrary field $\Bbbk$. All vector spaces, (co)algebras and bialgebras will be over $\Bbbk$.

Let **Vec** := category of vector spaces and **Bialg** := category of bialgebras.

Consider the following well-known adjunction

$$\begin{array}{ccc} \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\ {\scriptstyle T}\uparrow\downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P_1} \\ \mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 \end{array}$$

Here P sends a bialgebra B to its subspace of primitive elements

$$PB := \{b \in B \mid \Delta(b) = 1 \otimes b + b \otimes 1\},$$

while T sends a vector space V to the tensor algebra TV (which is indeed a bialgebra).

Vector spaces and bialgebras

Fix an arbitrary field $\Bbbk$. All vector spaces, (co)algebras and bialgebras will be over $\Bbbk$.

Let **Vec** := category of vector spaces and **Bialg** := category of bialgebras.

Consider the following well-known adjunction

$$\begin{array}{ccc} \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\ T \uparrow \downarrow P & & \downarrow P_1 \\ \mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 \end{array}$$

Here P sends a bialgebra B to its subspace of primitive elements

$$PB := \{b \in B \mid \Delta(b) = 1 \otimes b + b \otimes 1\},$$

while T sends a vector space V to the tensor algebra TV (which is indeed a bialgebra).
The unit $\eta V : V \to PTV$ of the adjunction is just the canonical inclusion.

Vector spaces and bialgebras

Fix an arbitrary field $\Bbbk$. All vector spaces, (co)algebras and bialgebras will be over $\Bbbk$.

Let **Vec** := category of vector spaces and **Bialg** := category of bialgebras.

Consider the following well-known adjunction

$$\begin{array}{ccc} \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\ T \uparrow \downarrow P & & \downarrow P_1 \\ \mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 \end{array}$$

Here P sends a bialgebra B to its subspace of primitive elements

$$PB := \{b \in B \mid \Delta(b) = 1 \otimes b + b \otimes 1\},$$

while T sends a vector space V to the tensor algebra TV (which is indeed a bialgebra).

The unit $\eta V : V \to PTV$ of the adjunction is just the canonical inclusion.
The counit $\varepsilon B : TPB \to B$ sends any tensor product of primitive elements to their product in B.

Now P_1 has a left adjoint T_1.

$$\begin{array}{ccc} \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\ T \uparrow \downarrow P & & T_1 \uparrow \downarrow P_1 \\ \mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 \end{array}$$

Now P_1 has a left adjoint T_1.

$$\begin{array}{ccc} \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\ T \uparrow \downarrow P & & T_1 \uparrow \downarrow P_1 \\ \mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 \end{array}$$

$\forall (V, \mu) \in \mathbf{Vec}_1$, by construction, $T_1(V, \mu)$ is defined to be the coequalizer

$$TPTV \underset{\varepsilon TV}{\overset{T\mu}{\rightrightarrows}} TV \overset{\pi}{\dashrightarrow} \frac{TV}{(z - \mu(z) | z \in PTV)} =: T_1(V, \mu) \, .$$

Now P_1 has a left adjoint T_1.

$$\begin{array}{ccc} \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\ T \uparrow \downarrow P & & T_1 \uparrow \downarrow P_1 \\ \mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 \end{array}$$

$\forall (V,\mu) \in \mathbf{Vec}_1$, by construction, $T_1(V,\mu)$ is defined to be the coequalizer

$$TPTV \underset{\varepsilon TV}{\overset{T\mu}{\rightrightarrows}} TV \xrightarrow{\pi} \frac{TV}{(z-\mu(z) \mid z \in PTV)} =: T_1(V,\mu)\ .$$

Now, note that $PTV = \oplus_{n \geq 1} P_n TV$ where $P_n TV := PTV \cap V^{\otimes n}$.

Now P_1 has a left adjoint T_1.

$$\begin{array}{ccc} \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\ T \uparrow \downarrow P & & T_1 \uparrow \downarrow P_1 \\ \mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 \end{array}$$

$\forall (V,\mu) \in \mathbf{Vec}_1$, by construction, $T_1(V,\mu)$ is defined to be the coequalizer

$$TPTV \underset{\varepsilon TV}{\overset{T\mu}{\rightrightarrows}} TV \xrightarrow{\pi} \frac{TV}{(z-\mu(z) \mid z \in PTV)} =: T_1(V,\mu)\ .$$

Now, note that $PTV = \oplus_{n\geq 1} P_n TV$ where $P_n TV := PTV \cap V^{\otimes n}$.
In particular, $P_1 TV = V$.

Now P_1 has a left adjoint T_1.

$$\begin{array}{ccc} \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\ T \uparrow \downarrow P & & T_1 \uparrow \downarrow P_1 \\ \mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 \end{array}$$

$\forall (V,\mu) \in \mathbf{Vec}_1$, by construction, $T_1(V,\mu)$ is defined to be the coequalizer

$$TPTV \underset{\varepsilon TV}{\overset{T\mu}{\rightrightarrows}} TV \overset{\pi}{\dashrightarrow} \frac{TV}{(z-\mu(z) \mid z \in PTV)} =: T_1(V,\mu) \,.$$

Now, note that $PTV = \oplus_{n\geq 1} P_n TV$ where $P_n TV := PTV \cap V^{\otimes n}$.
In particular, $P_1 TV = V$. Moreover $z - \mu(z) = 0$, for $z \in V$, so that

$$T_1(V,\mu) = \frac{TV}{(z-\mu(z) \mid z \in P_n TV, n \geq 2)}.$$

Now P_1 has a left adjoint T_1.

$$\begin{array}{ccc} \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\ T \uparrow \downarrow P & & T_1 \uparrow \downarrow P_1 \\ \mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 \end{array}$$

$\forall (V,\mu) \in \mathbf{Vec}_1$, by construction, $T_1(V,\mu)$ is defined to be the coequalizer

$$TPTV \overset{T\mu}{\underset{\varepsilon TV}{\rightrightarrows}} TV \xrightarrow{\pi} \frac{TV}{(z-\mu(z)|z\in PTV)} =: T_1(V,\mu)\ .$$

Now, note that $PTV = \oplus_{n\geq 1} P_n TV$ where $P_n TV := PTV \cap V^{\otimes n}$.
In particular, $P_1 TV = V$. Moreover $z - \mu(z) = 0$, for $z \in V$, so that

$$T_1(V,\mu) = \frac{TV}{(z-\mu(z) \mid z \in P_n TV, n \geq 2)}.$$

Consider now

$$S(V) := \frac{TV}{(z \mid z \in P_n TV, n \geq 2)} = \frac{TV}{(P_n TV \mid n \geq 2)}.$$

Now P_1 has a left adjoint T_1.

$$\begin{array}{ccc} \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\ T \uparrow \downarrow P & & T_1 \uparrow \downarrow P_1 \\ \mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 \end{array}$$

$\forall (V,\mu) \in \mathbf{Vec}_1$, by construction, $T_1(V,\mu)$ is defined to be the coequalizer

$$TPTV \underset{\varepsilon TV}{\overset{T\mu}{\rightrightarrows}} TV \overset{\pi}{\dashrightarrow} \frac{TV}{(z-\mu(z)|z\in PTV)} =: T_1(V,\mu)\ .$$

Now, note that $PTV = \oplus_{n\geq 1} P_n TV$ where $P_n TV := PTV \cap V^{\otimes n}$.
In particular, $P_1 TV = V$. Moreover $z - \mu(z) = 0$, for $z \in V$, so that

$$T_1(V,\mu) = \frac{TV}{(z-\mu(z) \mid z \in P_n TV, n \geq 2)}.$$

Consider now

$$S(V) := \frac{TV}{(z \mid z \in P_n TV, n \geq 2)} = \frac{TV}{(P_n TV \mid n \geq 2)}.$$

We have the following result.

Proposition

$$\mathrm{char}(\Bbbk) = 0 \quad \Longrightarrow \quad SV = \frac{TV}{(x \otimes y - y \otimes x \mid x, y \in V)}.$$

Proposition

$$\mathrm{char}(\Bbbk) = 0 \quad \Longrightarrow \quad SV = \frac{TV}{(x \otimes y - y \otimes x \mid x, y \in V)}.$$

$$\mathrm{char}(\Bbbk) = p > 0 \quad \Longrightarrow \quad SV = \frac{TV}{(x \otimes y - y \otimes x, x^p \mid x, y \in V)}.$$

Proposition

$$\mathrm{char}(\Bbbk) = 0 \implies SV = \frac{TV}{(x \otimes y - y \otimes x \mid x, y \in V)}.$$

$$\mathrm{char}(\Bbbk) = p > 0 \implies SV = \frac{TV}{(x \otimes y - y \otimes x, x^p \mid x, y \in V)}.$$

In both cases $PSV \cong V$.

Proof.

We sketch it for $\mathrm{char}(\Bbbk) = 0$ (the other case is similar).

Proposition

$$\mathrm{char}(\Bbbk) = 0 \quad \Longrightarrow \quad SV = \frac{TV}{(x \otimes y - y \otimes x \mid x, y \in V)}.$$

$$\mathrm{char}(\Bbbk) = p > 0 \quad \Longrightarrow \quad SV = \frac{TV}{(x \otimes y - y \otimes x, x^p \mid x, y \in V)}.$$

In both cases $PSV \cong V$.

Proof.

We sketch it for $\mathrm{char}(\Bbbk) = 0$ (the other case is similar).

Since $x \otimes y - y \otimes x \in P_2 TV, \forall x, y \in V$, there is a bialgebra projection

$$A := \frac{TV}{(x \otimes y - y \otimes x | x, y \in V)} \xrightarrow{\quad \gamma \quad} \frac{TV}{(P_n TV | n \geq 2)} = SV.$$

Proposition

$$\mathrm{char}(\mathbb{k}) = 0 \quad \Longrightarrow \quad SV = \frac{TV}{(x \otimes y - y \otimes x \mid x, y \in V)}.$$

$$\mathrm{char}(\mathbb{k}) = p > 0 \quad \Longrightarrow \quad SV = \frac{TV}{(x \otimes y - y \otimes x, x^p \mid x, y \in V)}.$$

In both cases $PSV \cong V$.

Proof.

We sketch it for $\mathrm{char}(\mathbb{k}) = 0$ (the other case is similar).

Since $x \otimes y - y \otimes x \in P_2 TV, \forall x, y \in V$, there is a bialgebra projection

$$A := \frac{TV}{(x \otimes y - y \otimes x | x, y \in V)} \xrightarrow{\quad\gamma\quad} \frac{TV}{(P_n TV | n \geq 2)} = SV.$$

It is well-known that $PA \cong V$.

Proposition

$$\mathrm{char}(\Bbbk) = 0 \quad \Longrightarrow \quad SV = \frac{TV}{(x \otimes y - y \otimes x \mid x, y \in V)}.$$

$$\mathrm{char}(\Bbbk) = p > 0 \quad \Longrightarrow \quad SV = \frac{TV}{(x \otimes y - y \otimes x, x^p \mid x, y \in V)}.$$

In both cases $PSV \cong V$.

Proof.

We sketch it for $\mathrm{char}(\Bbbk) = 0$ (the other case is similar).

Since $x \otimes y - y \otimes x \in P_2 TV, \forall x, y \in V$, there is a bialgebra projection

$$A := \frac{TV}{(x \otimes y - y \otimes x | x, y \in V)} \xrightarrow{\quad \gamma \quad} \frac{TV}{(P_n TV | n \geq 2)} = SV.$$

It is well-known that $PA \cong V$. Hence, $\gamma_{|PA}$ is injective.

Proposition

$$\mathrm{char}(\Bbbk) = 0 \quad \Longrightarrow \quad SV = \frac{TV}{(x \otimes y - y \otimes x \mid x, y \in V)}.$$

$$\mathrm{char}(\Bbbk) = p > 0 \quad \Longrightarrow \quad SV = \frac{TV}{(x \otimes y - y \otimes x, x^p \mid x, y \in V)}.$$

In both cases $PSV \cong V$.

Proof.

We sketch it for $\mathrm{char}(\Bbbk) = 0$ (the other case is similar).

Since $x \otimes y - y \otimes x \in P_2 TV, \forall x, y \in V$, there is a bialgebra projection

$$A := \frac{TV}{(x \otimes y - y \otimes x | x, y \in V)} \xrightarrow{\quad \gamma \quad} \frac{TV}{(P_n TV | n \geq 2)} = SV.$$

It is well-known that $PA \cong V$. Hence, $\gamma_{|PA}$ is injective.
By *Heyneman-Radford Theorem*, γ is injective whence bijective i.e. $SV = A$. □

We have the following result.

We have the following result.

Theorem

The adjunction (T_1, P_1) *is idempotent.*

We have the following result.

Theorem

The adjunction (T_1, P_1) is idempotent.

Proof.

Take $V_2 \in \mathbf{Vec}_2$.

We have the following result.

Theorem

The adjunction (T_1, P_1) is idempotent.

Proof.

Take $V_2 \in \mathbf{Vec}_2$. In particular we can write

$$V_2 = (V_1, P_1 T_1 V_1 \stackrel{\mu_1}{\to} V_1) \qquad \text{where} \qquad V_1 = (V, \mu) \in \mathbf{Vec}_1.$$

We have the following result.

Theorem

The adjunction (T_1, P_1) is idempotent.

Proof.

Take $V_2 \in \mathbf{Vec}_2$. In particular we can write

$$V_2 = (V_1, P_1 T_1 V_1 \stackrel{\mu_1}{\to} V_1) \qquad \text{where} \qquad V_1 = (V, \mu) \in \mathbf{Vec}_1.$$

Using that $\mu_1 \circ \eta_1 V_1 = \mathrm{Id}_{V_1}$ and $PS(V) \cong V$ one gets that

$$U_{0,1} \eta_1 V_1 : V \to U_{0,1} P_1 T_1 V_1 = P T_1 V_1$$

has inverse $U_{0,1}\mu_1$, where $U_{0,1} : \mathbf{Vec}_1 \to \mathbf{Vec}_0$.

We have the following result.

Theorem

The adjunction (T_1, P_1) is idempotent.

Proof.

Take $V_2 \in \mathbf{Vec}_2$. In particular we can write

$$V_2 = (V_1, P_1 T_1 V_1 \overset{\mu_1}{\to} V_1) \qquad \text{where} \qquad V_1 = (V, \mu) \in \mathbf{Vec}_1.$$

Using that $\mu_1 \circ \eta_1 V_1 = \mathrm{Id}_{V_1}$ and $PS(V) \cong V$ one gets that

$$U_{0,1} \eta_1 V_1 : V \to U_{0,1} P_1 T_1 V_1 = P T_1 V_1$$

has inverse $U_{0,1}\mu_1$, where $U_{0,1} : \mathbf{Vec}_1 \to \mathbf{Vec}_0$.
Since $U_{0,1}$ reflects isomorphisms, this implies $\eta_1 V_1 : V_1 \to P_1 T_1 V_1$ is an isomorphism.

We have the following result.

Theorem

The adjunction (T_1, P_1) is idempotent.

Proof.

Take $V_2 \in \mathbf{Vec}_2$. In particular we can write

$$V_2 = (V_1, P_1 T_1 V_1 \stackrel{\mu_1}{\to} V_1) \qquad \text{where} \qquad V_1 = (V, \mu) \in \mathbf{Vec}_1.$$

Using that $\mu_1 \circ \eta_1 V_1 = \mathrm{Id}_{V_1}$ and $PS(V) \cong V$ one gets that

$$U_{0,1}\eta_1 V_1 : V \to U_{0,1} P_1 T_1 V_1 = PT_1 V_1$$

has inverse $U_{0,1}\mu_1$, where $U_{0,1} : \mathbf{Vec}_1 \to \mathbf{Vec}_0$.
Since $U_{0,1}$ reflects isomorphisms, this implies $\eta_1 V_1 : V_1 \to P_1 T_1 V_1$ is an isomorphism.

Thus we conclude that $\eta_1 U_{1,2}$ is an isomorphism and we know this is equivalent to (T_1, P_1) idempotent. □

By the properties of idempotent adjunctions we have seen, we can complete the diagram to

$$\begin{array}{ccccc}
\mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\
T \uparrow \downarrow P & & T_1 \uparrow \downarrow P_1 & & T_2 \uparrow \downarrow P_2 \\
\mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 & \xleftarrow{U_{1,2}} & \mathbf{Vec}_2
\end{array}$$

where we can choose $T_2 := T_1 U_{1,2}$ and this is full and faithful.

By the properties of idempotent adjunctions we have seen, we can complete the diagram to

$$
\begin{array}{ccccc}
\mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\
T \uparrow \downarrow P & & T_1 \uparrow \downarrow P_1 & & T_2 \uparrow \downarrow P_2 \\
\mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 & \xleftarrow{U_{1,2}} & \mathbf{Vec}_2
\end{array}
$$

where we can choose $T_2 := T_1 U_{1,2}$ and this is full and faithful. In particular P has a monadic decomposition of length ≤ 2

By the properties of idempotent adjunctions we have seen, we can complete the diagram to

$$\begin{array}{ccccc}
\mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\
{\scriptstyle T}\uparrow\downarrow{\scriptstyle P} & & {\scriptstyle T_1}\uparrow\downarrow{\scriptstyle P_1} & & {\scriptstyle T_2}\uparrow\downarrow{\scriptstyle P_2} \\
\mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 & \xleftarrow{U_{1,2}} & \mathbf{Vec}_2
\end{array}$$

where we can choose $T_2 := T_1 U_{1,2}$ and this is full and faithful. In particular P has a monadic decomposition of length ≤ 2

By the foregoing, we know that $\mathrm{Im} P = \mathrm{Im} U_{0,2}$ so that the vector spaces arising as primitive part of some bialgebra are those isomorphic to the underlying vector space of some $V_2 \in \mathbf{Vec}_2$.

By the properties of idempotent adjunctions we have seen, we can complete the diagram to

$$\begin{array}{ccccc}
\mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\
T \uparrow \downarrow P & & T_1 \uparrow \downarrow P_1 & & T_2 \uparrow \downarrow P_2 \\
\mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 & \xleftarrow{U_{1,2}} & \mathbf{Vec}_2
\end{array}$$

where we can choose $T_2 := T_1 U_{1,2}$ and this is full and faithful. In particular P has a monadic decomposition of length ≤ 2

By the foregoing, we know that $\mathrm{Im} P = \mathrm{Im} U_{0,2}$ so that the vector spaces arising as primitive part of some bialgebra are those isomorphic to the underlying vector space of some $V_2 \in \mathbf{Vec}_2$.

Moreover we have $T_2 V_2 = T_1 U_{1,2} V_2 = T_1 V_1 = T_1(V, \mu)$.

By the properties of idempotent adjunctions we have seen, we can complete the diagram to

$$\begin{array}{ccccc}
\mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} & \xleftarrow{\mathrm{Id}_{\mathbf{Bialg}}} & \mathbf{Bialg} \\
T \uparrow \downarrow P & & T_1 \uparrow \downarrow P_1 & & T_2 \uparrow \downarrow P_2 \\
\mathbf{Vec} & \xleftarrow{U_{0,1}} & \mathbf{Vec}_1 & \xleftarrow{U_{1,2}} & \mathbf{Vec}_2
\end{array}$$

where we can choose $T_2 := T_1 U_{1,2}$ and this is full and faithful. In particular P has a monadic decomposition of length ≤ 2

By the foregoing, we know that $\mathrm{Im} P = \mathrm{Im} U_{0,2}$ so that the vector spaces arising as primitive part of some bialgebra are those isomorphic to the underlying vector space of some $V_2 \in \mathbf{Vec}_2$.

Moreover we have $T_2 V_2 = T_1 U_{1,2} V_2 = T_1 V_1 = T_1(V, \mu)$.

Indeed we can be more precise....

Theorem

Take $V_2 := ((V,\mu),\mu_1) \in \mathbf{Vec}_2$.

Theorem

Take $V_2 := ((V,\mu),\mu_1) \in \mathbf{Vec}_2$.
$\mathrm{char}\Bbbk = 0$) *We have that* $(V,[-,-])$ *is a Lie algebra where*

$$[-,-]: V \otimes V \to V, \qquad [x,y] := \mu(xy - yx).$$

Theorem

Take $V_2 := ((V,\mu),\mu_1) \in \mathbf{Vec}_2$.
$\mathrm{char}\Bbbk = 0$) *We have that* $(V,[-,-])$ *is a Lie algebra where*

$$[-,-] : V \otimes V \to V, \qquad [x,y] := \mu(xy - yx).$$

Moreover $T_2 V_2$ *is the universal enveloping algebra*

$$T_2 V_2 = \frac{TV}{(xy - yx - [x,y] \mid x,y \in V)}.$$

Theorem

Take $V_2 := ((V,\mu),\mu_1) \in \mathbf{Vec}_2$.

$\mathrm{char}\Bbbk = 0$) *We have that* $(V,[-,-])$ *is a Lie algebra where*

$$[-,-]: V \otimes V \to V, \qquad [x,y] := \mu(xy - yx).$$

Moreover $T_2 V_2$ *is the universal enveloping algebra*

$$T_2 V_2 = \frac{TV}{(xy - yx - [x,y] \mid x,y \in V)}.$$

$\mathrm{char}\Bbbk = p$)

We have that $\left(V,[-,-],-^{[p]}\right)$ *is a restricted Lie algebra where*

$$[-,-]: V \otimes V \to V, [x,y] := \mu(xy - yx) \quad \text{and} \quad -^{[p]}: V \to V, x^{[p]} := \mu(x^p).$$

Theorem

Take $V_2 := ((V,\mu),\mu_1) \in \mathbf{Vec}_2$.

$\mathrm{char}\Bbbk = 0$) *We have that* $(V,[-,-])$ *is a Lie algebra where*

$$[-,-] : V \otimes V \to V, \qquad [x,y] := \mu(xy - yx).$$

Moreover T_2V_2 *is the universal enveloping algebra*

$$T_2V_2 = \frac{TV}{(xy - yx - [x,y] \mid x,y \in V)}.$$

$\mathrm{char}\Bbbk = p$)

We have that $\left(V,[-,-],-^{[p]}\right)$ *is a restricted Lie algebra where*

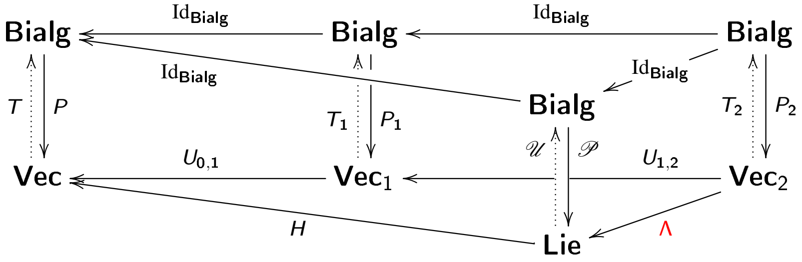
$$[-,-] : V \otimes V \to V, [x,y] := \mu(xy - yx) \quad \text{and} \quad -^{[p]} : V \to V, x^{[p]} := \mu(x^p).$$

Moreover T_2V_2 *is the restricted enveloping algebra*

$$T_2V_2 = \frac{TV}{\left(xy - yx - [x,y], x^p - x^{[p]} \mid x,y \in V\right)}.$$

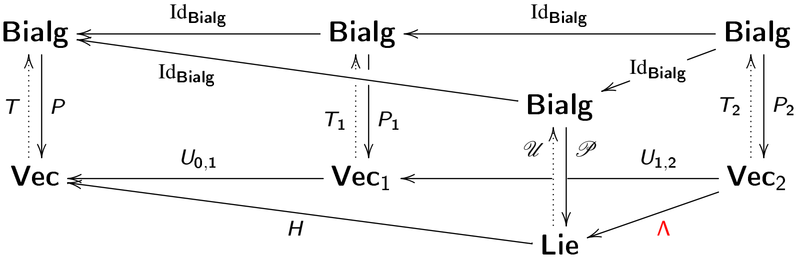
Work in progress

As a consequence, in a work in progress with I. Goyvaerts and C. Menini, we prove that there is an equivalence of categories Λ such that $\Lambda \circ P_2 = \mathscr{P}$ and $H \circ \Lambda = U_{0,2}$ where



Work in progress

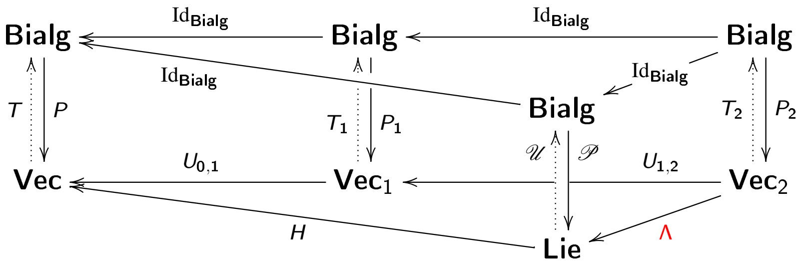
As a consequence, in a work in progress with I. Goyvaerts and C. Menini, we prove that there is an equivalence of categories Λ such that $\Lambda \circ P_2 = \mathscr{P}$ and $H \circ \Lambda = U_{0,2}$ where



Here **Lie** denotes either the category of Lie algebras or the category of restricted Lie algebras depending on the characteristic, and $\mathscr{U}$ is the corresponding universal enveloping algebra.

Work in progress

As a consequence, in a work in progress with I. Goyvaerts and C. Menini, we prove that there is an equivalence of categories Λ such that $\Lambda \circ P_2 = \mathscr{P}$ and $H \circ \Lambda = U_{0,2}$ where



Here **Lie** denotes either the category of Lie algebras or the category of restricted Lie algebras depending on the characteristic, and $\mathscr{U}$ is the corresponding universal enveloping algebra.

Thus $\mathbf{Vec}_2 \cong$ **Lie** so that monadic decomposition leads to **Lie**.