Uniform Artin-Rees Properties for syzygies

Janet Striuli,
Joint work with
Ian Aberbach and Aline Hosry

Department of Mathematics, Fairfield University

ASTA 2014– *Spineto*

The Artin-Rees Lemma

Let $N \subset M$ two modules over a noetherian ring R and let I an ideal. Then there exists an integer h such that

$$I^n M \cap N \subseteq I^{n-h} N,$$

for all $n \geq h$.

The Uniform Artin-Rees Lemma

Question: (Eisenbud-Hochster)

Given an inclusion of R-modules $N \subseteq M$, does it exist an integer h such that

$$\mathfrak{m}^n M \cap N \subseteq \mathfrak{m}^{n-h} N,$$

for all maximal ideals $\mathfrak{m}$ of R?

Theorem: (Duncan–O'Carroll)

The previous question has a positive answer if the ring is excellent.

The Uniform Artin-Rees Lemma over local rings

Theorem: (Huneke)

Let R be a local ring. Given an inclusion of R-modules, there exists an integer h such that

$$I^n M \cap N \subseteq I^{n-h} N,$$

for all $n \geq h$ and for all $I \subset R$.

R satisfying the above Theorem is said to have the uniform Artin-Rees property.

Ideas in the proof of the Uniform Artin-Rees property.

Theorem (Huneke):

Let R be a noetherian ring such that for every prime ideal $\mathfrak{p}$:

- there exists an element $0 \neq c \in R/\mathfrak{p}$ such that c annihilates the homology of complexes that satisfy the standard conditions on rank and height;
- there exists an element $0 \neq c \in R/\mathfrak{p}$ and an integer k such that $c\overline{I^n} \subseteq I^{n-k}$ for all ideals I and $n \geq k$,

then R has the uniform Artin-Rees property.

Ideas in the proof of the Uniform Artin-Rees property.

Complexes satisfying the standard conditions.

Let

$$G_\bullet : 0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to 0,$$

where $\partial_i : G_i \to G_{i-1}$ be a complex of finitely generated free modules. $G_\bullet$ satisfies:

- the standard conditions on rank if $\operatorname{rank}(\partial_i) + \operatorname{rank}(\partial_{i-1}) = \operatorname{rank} G_i$;
- the standard conditions on height if $\operatorname{ht}_R I(\partial_i) \geq i$.

Idea in the proof of Uniform Artin-Rees property: the role of Koszul Complex.

Theorem (Huneke):

Let R be an arbitrary commutative ring and let $M_\bullet$ be a complex

$$0 \to M_n \to \cdots \to M_1 \to M_0 \to 0$$

of arbitrary R-modules. Let $\underline{x} = x_1, \ldots, x_n$ be a sequence of elements in R, and let $d, d_0, d_1, \ldots, d_{n-2}$ be elements of R such that

1. d_i kills $H_{n-i}(M_\bullet)$, for $0 \leqslant i \leqslant n-2$ (this is where the correction is), and
2. d kills $H_{n-j}(x_1, \ldots, x_n; M_{j+1})$, for $1 \leqslant j \leqslant n-1$.

Then $D = (d_0 d_1 \ldots d_{n-2}) d^n$ kills $\mathrm{Hom}_R(R/(x_1, \ldots, x_n), H_1(M_\bullet))$.

Artin-Rees for syzygies

Let $(R, \mathfrak{m}, k)$ be a local ring.
Let M be a finitely generated R-module.
Let $(\mathbf{F}_\bullet, \partial_\bullet)$ be a free resolution of M:

$$\mathbf{F}_\bullet : \cdots \to F_i \to F_{i-1} \to \cdots \to F_0 \to M$$

with $\partial_i : F_i \to F_{i-1}$

Question (Eisenbud-Huneke):

Fix an ideal $I \subseteq R$ and an R-module M. Consider a free resolution of M, $(\mathbf{F}_\bullet, \partial_\bullet)$. Is there an integer h such that

$$I^n F_i \cap \ker \partial_i \subseteq I^{n-h} \ker \partial_i$$

for all $n \geq h$ and for all $i \geq 1$?

Artin-Rees for syzygies

Let $(R, \mathfrak{m}, k)$ be a local ring.
Let M be a finitely generated R-module.
Let $(\mathbf{F}_\bullet, \partial_\bullet)$ be a free resolution of M:

$$\mathbf{F}_\bullet : \cdots \to F_i \to F_{i-1} \to \cdots \to F_0 \to M$$

with $\partial_i : F_i \to F_{i-1}$

Question (Eisenbud-Huneke):

Fix an ideal $I \subseteq R$ and an R-module M. Consider a free resolution of M, $(\mathbf{F}_\bullet, \partial_\bullet)$. Is there an integer h such that

$$I^n F_i \cap \ker \partial_i \subseteq I^{n-h} \ker \partial_i$$

for all $n \geq h$ and for all $i \geq 1$?

Definition

If the answer to the above question is positive then we say that M is syzygetically Artin-Rees with respect to I.

Theorem (Eisenbud-Huneke)

Let M be an R-module which is free in the punctured spectrum and let I be an $\mathfrak{m}$-primary ideal. Then, M is syzygetically Artin-Rees with respect to I.

Other Examples:

- (Levin) k is syzygetically Artin-Rees with respect of $\mathfrak{m}$. In fact

$$\mathfrak{m}^n F_i \cap \partial_i = \mathfrak{m}^{n-h}(\mathfrak{m}^h F_i \cap \partial_i).$$

- Let I be a principal ideal, then every module M is syzygetically Artin-Rees with respect to I.
- if R is Cohen-Macaulay, then every R-module M is syzygetically Artin-Rees with respect to an $\mathfrak{m}$-primary ideal.
- (–) Let R be a noetherian ring of dimension at most 2 and I an $\mathfrak{m}$-primary ideal. Then every R-module is syzygetically Artin-Rees with respect to I.

Other Examples:

- (Levin) k is syzygetically Artin-Rees with respect of $\mathfrak{m}$. In fact

$$\mathfrak{m}^n F_i \cap \partial_i = \mathfrak{m}^{n-h}(\mathfrak{m}^h F_i \cap \partial_i).$$

- Let I be a principal ideal, then every module M is syzygetically Artin-Rees with respect to I.
- if R is Cohen-Macaulay, then every R-module M is syzygetically Artin-Rees with respect to an $\mathfrak{m}$-primary ideal.
- (–) Let R be a noetherian ring of dimension at most 2 and I an $\mathfrak{m}$-primary ideal. Then every R-module is syzygetically Artin-Rees with respect to I.

Other Examples:

- (Levin) k is syzygetically Artin-Rees with respect of $\mathfrak{m}$. In fact

$$\mathfrak{m}^n F_i \cap \partial_i = \mathfrak{m}^{n-h}(\mathfrak{m}^h F_i \cap \partial_i).$$

- Let I be a principal ideal, then every module M is syzygetically Artin-Rees with respect to I.
- if R is Cohen-Macaulay, then every R-module M is syzygetically Artin-Rees with respect to an $\mathfrak{m}$-primary ideal.
- (–) Let R be a noetherian ring of dimension at most 2 and I an $\mathfrak{m}$-primary ideal. Then every R-module is syzygetically Artin-Rees with respect to I.

Other Examples:

- (Levin) k is syzygetically Artin-Rees with respect of $\mathfrak{m}$. In fact

$$\mathfrak{m}^n F_i \cap \partial_i = \mathfrak{m}^{n-h}(\mathfrak{m}^h F_i \cap \partial_i).$$

- Let I be a principal ideal, then every module M is syzygetically Artin-Rees with respect to I.
- if R is Cohen-Macaulay, then every R-module M is syzygetically Artin-Rees with respect to an $\mathfrak{m}$-primary ideal.
- (–) Let R be a noetherian ring of dimension at most 2 and I an $\mathfrak{m}$-primary ideal. Then every R-module is syzygetically Artin-Rees with respect to I.

Annihilators of Homology

An obvious connection

Let I be an ideal of R and M a finitely generated R module. Then

$$\operatorname{Tor}_i^R(M, R/I^n) = \frac{\operatorname{Im}\partial_i \cap I^n F_{i-1}}{I^n \operatorname{Im}\partial_i}$$

for all $i \geq 0$ and for all $n \geq 0$. So if there exists an integer h such that

$$I^h \operatorname{Tor}_i^R(M, R/I^n) = 0$$

for all $n \geq h$ and $i \geq 0$, then M is syzygetically Artin-Rees with respect to the ideal I.

Koszul Annihilating Sequences

Definition:

Let $(R, \mathfrak{m})$ be a local ring of dimension d and let M be a finitely generated R-module.
A sequence of elements, $c_1, \ldots, c_d \in \mathfrak{m}$ is a KAS for M if

1. the elements $c_1, \ldots, c_d$ are a system of parameters for R, and
2. for all integers k such that $d \geqslant k \geqslant i-1$, if $x_1, \ldots, x_k, c_{k+1}, \ldots, c_d$ are a system of parameters for R, then the equality

$$c_v H_n(x_1, \ldots, x_k, c_{k+1}^t, \ldots, c_j^t; M) = 0$$

   holds for all $v \geqslant j \geqslant k \geqslant 1$ and for all $n, t \geqslant 1$.

If $\mathcal{M}$ is a family of finitely generated R-modules, and $c_i, \ldots, c_d$ is a *KAS* for all $M \in \mathcal{M}$, then we will say that the sequence is a *KAS* for $\mathcal{M}$.

Remark:

Over a local ring R KAS sequences exists.

Well-suited sequences.

Definition:

Let $\underline{c} = c_1, \dots, c_d$ be a KAS for a family of modules $\mathcal{M}$. A system of parameters $\underline{x} = x_1, \dots, x_d$ is said to be *well-suited to* $\underline{c}$ if for all $1 \leqslant i \leqslant j \leqslant d$, any subset of cardinality $d - (j - i + 1)$ of the sequence $\underline{x}$ together with the sequence $c_i, \dots, c_j$ is a system of parameters for R.

Remark:

Let $\underline{c} = c_1, \dots, c_d$ be a *kas* sequence for the family of modules which are dth syzygies and let $x_1, \dots, x_d$ be a well-suited sequence to $\underline{c}$. Then

$$c_v H_t(x_1, \dots, x_k, c_{k+1}, \dots, c_\ell; M) = 0,$$

$$c_v H_t(x_1, \dots, x_k, c_k, \dots, c_{\ell-1}; M) = 0,$$

for all integers $v \geqslant \ell \geqslant k \geqslant 1$, for all $t \geqslant 1$, and for all R-modules M that are dth syzygies.

Theorem:

Let $(R, \mathfrak{m})$ be a local ring of dimension d and let $c_1, \ldots, c_d$ be a KAS for the family of modules which are dth syzygies. Then there exists an integer $t \geqslant 0$ such that

$$c_{n+j}^t H_i(G_\bullet \otimes M) = 0,$$

for all integers i and j such that $i \geqslant 1$, $0 \leqslant j \leqslant d - n$, for every dth syzygy M, and for every complex $(G_\bullet, \partial_\bullet)$ of length $n \leqslant d$ of finitely generated free R-modules with the following properties:

1. $G_\bullet$ satisfies the standard conditions on rank,
2. For $1 \leqslant i \leqslant n$, the ideal $I(\partial_i) + (c_{i+1}, \ldots, c_d)$ is $\mathfrak{m}$-primary.

The Eagon-Northcott complex

Let $\underline{x} = x_1, \ldots, x_h$ be a sequence elements of a ring R.
Consider the $n \times (n+h-1)$ matrix

$$\mathcal{B} = \begin{pmatrix} x_1 & x_2 & \ldots & x_h & 0 & \ldots & \ldots & 0 \\ 0 & x_1 & x_2 & \ldots & x_h & 0 & \ldots & 0 \\ \ldots & & & & & & & \\ 0 & 0 & \ldots & \ldots & & & & x_h \end{pmatrix}.$$

The $n \times n$ minors of $\mathcal{B}$ are the generators of the ideal J^n, where $J = (x_1, \ldots, x_h)$. Eagon and Northcott construct a complex $(B_\bullet^{J,n}, \partial_\bullet)$ such that $H_0(B_\bullet^{J,n}) = R/J^n$.

Reductions

Let $(R, \mathfrak{m}, k)$ be a complete local ring. There exists a positive integer k such that for all ideals I and all reductions $J \subseteq I$, the inclusion $I^n \subseteq J^{n-k}$ holds for all $n \geqslant k$.

Definition:

Let I be an $\mathfrak{m}$-primary ideal and $\underline{c} = c_1, \ldots, c_d$ be a *kas* sequence. A special reduction of I with respect to $\underline{c}$ is a sequence $x_1, \ldots, x_d$ which is well-suited to sequence of c_i and verifies the following, for all integers i such that $0 \leqslant i \leqslant d-1$:

1. I_{d-i-1} is a reduction of I_{d-i} modulo $(c_{d-i}, \ldots, c_d)$
2. I_{d-i-1} is a reduction of I_{d-i} modulo $(0 : (0 : c_{d-i}))$,

where I_k denotes the ideal generated by $x_1, \ldots, x_k$. We also set $I_0 = 0$.

The Main Reduction Step:

Theorem:

Let $(R, \mathfrak{m})$ be a local ring of dimension d with an infinite residue field. Let $\underline{c} = c_1, \ldots, c_d$ be a *KAS*-sequence of R, and let $0 \to N \to G \to M \to 0$ be a short exact sequence of finitely generated R-modules where M is a dth syzygy and G is a free R-module. There exists an integer h, depending only on $\underline{c}$, such that for any sequence $x_1, \ldots, x_d$ which is a special reduction, and for all integers i such that $0 \leqslant i \leqslant d-1$ and for all $n \geqslant h$, the following inclusion holds

$$I_{d-i}^{n} G \cap N \subseteq I_{d-i}^{n-h} N + I_{d-i-1}^{n-h} G \cap N,$$

where $I_j = (x_1, \ldots, x_j)$.

Theorem:

Let $(R, \mathfrak{m}, k)$ be a complete local ring. There exists a positive integer k such that for all ideals I and all reductions $J \subseteq I$, the inclusion $I^n \subseteq J^{n-k}$ holds for all $n \geqslant k$.

Theorem:

Let $(R, \mathfrak{m})$ be a local noetherian ring of dimension d. The family of modules that are d-syzygies are syzygetically Artin-Rees with respect to the family of *all* the ideals of R.

Theorem:

Let $(R, \mathfrak{m})$ be a local noetherian ring. Every finitely generated R-module M is syzygetically Artin-Rees with respect to the family of all ideals of R.