Power Graphs of Rings

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June 16, 2014

Outline

1. Graph Survey
2. The Power Graph
3. Ring-Graph Connections

Graphs Defined on a Ring R

Zero Divisor Graph

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Unit Graph

Defined on a ring with unit $1 \neq 0$. $V = R$ and $(a, b) \in E$ if and only if $a + b$ is a unit.

More Graphs on the Ring R

Generalized Total Graph

Defined on a commutative ring. Let S be a multiplicatively closed subset of R. Define the vertices of the graph to be the elements of R and $(a, b) \in E$ if and only if $a + b \in S$. If S is the set of zero divisors union $\{0\}$ then this is the Total Graph. If S is the set of units of R, this is the Unit Graph.

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The ring R is commutative. The vertices are elements of R and $(a, b) \in E$ if and only if $Ra + Rb = R$.

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Jacobson Graph

The ring R is commutative with nonzero identity. Let $J(R)$ be the Jacobson Radical of R. The vertex set $V = R - J(R)$ and $(a, b) \in E$ if and only if $1 - ab$ is not invertible.

Graphs Using Ideals

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Intersection Graph

The vertices are nontrivial left ideals of R. Two ideals I and J are connected by an edge if $I \cap J \neq 0$.

Why another graph on a ring?

My Motivation for Graph Theory

- Let R be a CDVD with prime p and quotient field Q
- Let $T = \oplus_{i\geq 1}\langle b_i\rangle, \quad o(b_i) = p^i$
- Define M_α as the extension

$$E_\alpha\colon 0 \longrightarrow T \longrightarrow M_\alpha \longrightarrow Q \longrightarrow 0$$

- (May) If $End(M_\alpha \cong End(M_\beta)$, then $M_\alpha \cong M_\beta$
- (Flagg) If $J(E(M_\alpha)) \cong J(E(M_\beta))$, then $M_\alpha \cong M_\beta$
- I have a mental picture of which endomorphisms are in or not in the endomorphism ring, how can I quantify that?

Reality Graph Theory

- The idempotent elements of an endomorphism ring connect the ring to the underlying group or module.
- Multiplicative orders of endomorphisms tells us something about the underlying group or module.
- The combination of idempotents and zero divisors also tells us something about the group.

Power Graph of a Semigroup

Let $(S, *)$ be a semigroup.

Directed Power Graph

The directed power graph of S, $\overrightarrow{\mathcal{G}(S)}$ is the graph with vertex set S and given two vertices, a and b, there is an arc $a \longrightarrow b$ if and only if $a \neq b$ and $b = a^m$ for some positive integer m.

Undirected Power Graph

The (undirected) power graph of a semigroup S, $\mathcal{G}(S)$, is the graph without any arrows or multiple edges. The vertices a and b are connected by an edge if and only if $a \neq b$ and either $a = b^n$ or $b = a^m$ for some positive integers m and n.

Properties of the Power Graph

For a finite semigroup S

- $\overrightarrow{\mathcal{G}(S)}$, $(\mathcal{G}(S))$, is connected if and only if S has exactly one idempotent.
- Each connected component contains exactly one idempotent.

For an infinite semigroup, the graph is generally not connected, and connected components do not necessarily contain an idempotent. However, a vertex is a sink in $\overrightarrow{\mathcal{G}(S)}$ if and only if it is an idempotent.

Power Graphs of Finite Groups

- If A and B are finite groups (not necessarily abelian) such that their (undirected) power graphs are isomorphic, then A and B have the same number of elements of each order.
- The graph is complete if and only if A is a cyclic group of order p^m for some prime p and positive integer m.
- If A and B are finite abelian groups such that their undirected power graphs are isomorphic, then $A \cong B$.

The Power Graphs of Rings

Define

Multiplicative Power Graph

The (directed) multiplicative power graph of the ring R, denoted $\mathcal{G}^{\times}(R)$ or $\overrightarrow{\mathcal{G}^{\times}(R)}$, is the power graph of the semigroup $(R, \times)$.

Additive Power Graph

The (directed) additive power graph of the ring R, denoted $\mathcal{G}^{+}(R)$ or $\overrightarrow{\mathcal{G}^{+}(R)}$, is the power graph of the abelian group $(R, +)$.

Note that if R is finite, its additive power graph determines the additive structure of the ring.

Basic Connections

Let R be a finite ring with unity $1 \neq 0$:

- $\mathcal{G}^{\times}(R)$ has at least two connected components.
- If $\mathcal{G}^{\times}(R)$ has only two connected components, then R is indecomposable.
- if $\mathcal{G}^{\times}(R)$ has exactly 2 components, one an isolated vertex, then R is a field.

If $\mathcal{G}^{\times}(R)$ is connected, R does not have a unity.
Let $R = End(A)$ for some finite abelian group A.

- $\mathcal{G}^{\times}(R)$ has at least two connected components.
- If $\mathcal{G}^{\times}(R)$ has exactly two components, then A is indecomposable.

Rings of Square-free order

Theorem

If R is a ring of order n, a square-free integer, and S is a ring such that $\mathcal{G}^{\times}(R) \cong \mathcal{G}^{\times}(S)$, then $R \cong S$.

Proof

$n = p_1 p_2 p_3 \cdots p_k$. The ring $R = R_1 \times R_2 \times \cdots \times R_k$ where $|R_i| = p_i$ There are 2^k possible rings (only one with unity). Each summand R_i is either $\mathbb{Z}_{p_i}$ with trivial multiplication or with the normal multiplication mod p. The graph $\mathcal{G}^{\times}(R)$ has 2^t connected components, where t is the number of summands with nontrivial multiplication. The cardinality of the smallest component is the product of the primes p_i for which R_i has the trivial multiplication.

Rings of Order p^2

There are 11 nonisomorphic rings of order p^2 for any prime p.
Three are rings on the additive group $\mathbb{Z}_{p^2}$ and 8 are on the additive ring $\mathbb{Z}_p \times \mathbb{Z}_p$.
Four of these ring have a unity, 7 do not.
Nine of these rings are commutative, and two are not. The two noncommutative rings are opposite rings of each other.

Theorem

Let R is a ring of order p^2, for some prime $p \geq 3$. Suppose S is a ring such that $\mathcal{G}^{\times}(R) \cong \mathcal{G}^{\times}(S)$ and $\mathcal{G}^{+}(R) \cong \mathcal{G}^{+}(S)$. Then $S \cong R$ or $S \cong R^{op}$.

Rings of Order 4

The possible rings:

- $A = \langle a \colon 4a = 0, a^2 = a \rangle$
- $B = \langle a \colon 4a = 0, a^2 = 2a \rangle$
- $C = \langle a \colon 4a = 0, a^2 = 0 \rangle$
- $D = \langle a, b \colon 2a = 2b = 0, a^2 = a, b^2 = b, ab = ba = 0 \rangle$
- $E = \langle a, b \colon 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b \rangle$
- $F = \langle a, b \colon 2a = 2b = 0, a^2 = a, b^2 = b, ab = b, ba = a \rangle$
- $G = \langle a, b \colon 2a = 2b = 0, a^2 = 0, b^2 = b, ab = a, ba = a \rangle$
- $H = \langle a, b \colon 2a = 2b = 0, a^2 = 0, b^2 = b, ab = ba = 0 \rangle$
- $I = \langle a, b \colon 2a = 2b = 0, a^2 = b, ab = 0 \rangle$
- $J = \langle a, b \colon 2a = 2b = 0, a^2 = b^2 = 0 \rangle$
- $K = GF(4)$ the field with 4 elements

Multiplicative Power Graphs of Order 4

Ring	Additive Group	Multiplicative Power Graph
A	$\mathbb{Z}_4$	
G	$\mathbb{Z}_2 \times \mathbb{Z}_2$	
H	$\mathbb{Z}_2 \times \mathbb{Z}_2$	
B	$\mathbb{Z}_4$	
I	$\mathbb{Z}_2 \times \mathbb{Z}_2$	
C	$\mathbb{Z}_4$	
J	$\mathbb{Z}_2 \times \mathbb{Z}_2$	
E	$\mathbb{Z}_2 \times \mathbb{Z}_2$	
F	$\mathbb{Z}_2 \times \mathbb{Z}_2$	
D	$\mathbb{Z}_2 \times \mathbb{Z}_2$	
K	$\mathbb{Z}_2 \times \mathbb{Z}_2$	

Cyclic Rings of Order p^m

Theorem

Let R be a ring with $(R, +)$ cyclic of order p^m. Suppose S is a ring such that $\mathcal{G}^\times(R) \cong \mathcal{G}^\times(S)$ and $\mathcal{G}^+(R) \cong \mathcal{G}^+(S)$. Then, $R \cong S$.

Where Next?

- This is just the beginning of the investigation, and a promising one.
- The counting techniques used here should lead to a more general equivalence relation argument, I hope.
- Matrix rings, including endomorphism rings, are rich in idempotents, and I have barely started!
- The combination of the multiplicative power graph and the zero divisor graph has great potential to see ring properties in graphical form.
- I want to consider group rings.
- The power graph of an infinite ring is very disconnected, however I wonder if the combination of the multiplicative power graph and the zero divisor graph says anything about algebraic entropy?

Thanks

I'd like to thank the organizing committee for their hard work, and all involved for planning a conference in this beautiful location!

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