Gorenstein homological algebra relative to weakly Wakamatsu (co)tilting modules

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Outline of the talk

This is a joint work with Driss Bennis and Luis Oyonarte.

- Introduction. Weakly Wakamatsu tilting modules.
- G_C-projective modules.
- Relations with Auslander and Bass classes
- G_C-projective dimension.
- $\mathscr{P}_C$-projective dimension.

Given the class $\mathcal{F}$, we denote

$$\mathcal{F}^{\perp} = \{N;\ \mathrm{Ext}_R^{\geq 1}(F, N) = 0\ \forall F \in \mathcal{F}\},$$

$$^{\perp}\mathcal{F} = \{N;\ \mathrm{Ext}_R^{\geq 1}(N, F) = 0\ \forall F \in \mathcal{F}\}.$$

A left $\mathcal{F}$-resolution of an R-module M is a complex

$$\mathbf{X} = \cdots \to X_1 \to X_0 \to M \to 0,$$

such that $X_i \in \mathcal{F}$ and $\mathrm{Hom}_R(F, \mathbf{X})$ is an exact complex for every $F \in \mathcal{F}$. Dually, right $\mathcal{F}$-resolutions are defined.

We say that $M \in R - Mod$ is Gorenstein projective if M is a syzygy of an exact complex of projectives $X^{\bullet}$ which verifies that $Hom_R(X^{\bullet}, Q)$ is also exact for every projective $Q \in R - Mod$.

We denote by $GP(R)$ (resp. $P(R)$) the class of Gorenstein projective (resp. projective) left R-modules.

It is clear that $P(R) \subseteq GP(R)$

Definition

We call a class of left R-modules $\mathcal{X}$ projectively resolving if $P(R) \subseteq \mathcal{X}$, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$ the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent.

Definition (Bennis, Oyonarte, Rozas)

A left R-module C is weakly Wakamatsu tilting (w-tilting for short) if it has the following two properties:

1. $\mathrm{Ext}_R^{\geq 1}(C, C^{(I)}) = 0$ *for every set* I.
2. *There exists an exact sequence of left R-modules*

$$\mathbf{X}: \; 0 \longrightarrow R \xrightarrow{f_0} C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} \cdots$$

where, for every $i \in \mathbb{N}$, $C_i \in Add(C)$ and such that $\mathrm{Hom}_R(-, E)$ leaves the sequence $\mathbf{X}$ exact whenever $E \in Add_R(C)$.

If C satisfies 1. but perhaps not 2. then C will be said to be Σ-self-orthogonal.

Definition

Given any $C \in R$-Mod, an R-module M is said to be G$_C$*-projective if there exists an exact sequence of R-modules*

$$\mathbf{X} = \ \cdots \to P_1 \to P_0 \to A^0 \to A^1 \to \cdots$$

where the $P_i' s$ are all projective, $A^i \in Add_R(C)$ for every $i \in \mathbb{N}$, $M \cong \mathrm{Im}(P_0 \to A^0)$, and such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence $\mathbf{X}$ exact whenever $Q \in Add_R(C)$.
We use $G_C P(R)$ to denote the class of all G$_C$*-projective R-modules.*

Proposition

Let C be an R-module. An R-module M is G$_C$-projective if and only if

1. $M \in {}^{\perp}Add_R(C)$.
2. *There exists an exact sequence* $\mathbf{X} = \ 0 \to M \to A^0 \to A^1 \to \cdots$, *where* $(A^i)'s \in Add_R(C)$, *such that* $\mathrm{Hom}_R(-, Q)$ *leaves the sequence* $\mathbf{X}$ *exact whenever* $Q \in Add_R(C)$.

Proposition

Let C be an R-module. The following conditions are equivalent:

1. $C \in G_C P(R)$.
2. C *is* Σ*-self-orthogonal.*
3. $Add_R(C) \subseteq Add_R(C)^{\perp}$.

Also, under the condition $C \in G_C P(R)$, we have the following characterization of G_C-projective modules.

Proposition

Let C be a Σ-self-orthogonal R-module. Then an R-module M is G_C-projective if and only if there exists an exact sequence of R-modules $0 \to M \to A \to G \to 0$ where $A \in Add_R(C)$ and G is G_C-projective.

Proposition

Let C be a Σ-self-orthogonal R-module. Then, the following conditions are equivalent:

1. C *is w-tilting.*
2. $R \in G_C P(R)$.
3. $P(R) \subseteq G_C P(R)$.

Therefore, when C is w-tilting every left $G_C P(R)$-resolution is exact.

Theorem

1) Let C be any R-module. Then the class $G_CP(R)$ is closed under direct sums and extensions.
2) Let C be a Σ-self-orthogonal R-module. Then the class $G_CP(R)$ is closed under kernels of epimorphisms and under direct summands.
3) An R-module C is w-tilting if and only if G$_C P(R)$ contains C and it is projectively resolving.

Corollary

Let C be a Σ-self-orthogonal R-module. Then, for every exact sequence of R-modules

$$\mathbf{X} = \ \cdots \to P_1 \to P_0 \to A^0 \to A^1 \to \cdots$$

where the P_i's are all projective and $A^i \in Add_R(C)$ for every $i \in \mathbb{N}$, if $\mathrm{Hom}_R(-, Q)$ leaves the sequence $\mathbf{X}$ exact whenever $Q \in Add_R(C)$, then every $\mathrm{Ker}(A^i \to A^{i+1})$ is G_C-projective. If, in addition, C be a w-tilting R-module, then every $\mathrm{Im}(P_{i+1} \to P_i)$ is also G_C-projective.

Corollary

Let C be a w-tilting R-module. The following conditions are equivalent.
i) $Add_R(C) \subseteq GP(R)^{\perp}$.
ii) $GP(R) \subseteq G_C P(R)$.
In particular, if $inj.dim_R(C) < \infty$ or $proj.dim_R(C) < \infty$ then $GP(R) \subseteq G_C P(R)$.

Proposition

If C is w-tilting then $GP(R) = G_C P(R)$ if and only if C is a projective generator of R-Mod (that is, $Add_R(C) = P(R)$).

For a class $\mathcal{F} \subseteq R - Mod$, let $\overline{\mathcal{F}}$ be the class of left R-modules with finite left $\mathcal{F}$-dimension.

Proposition

Let C be a w-tilting R-module such that $GP(R) \subseteq G_C P(R)$ and consider the assertions

1. $P(R) = Add_R(C) \cap \overline{P}(R)$.
2. $GP(R) = G_C P(R) \cap \overline{GP}(R)$.
3. $P(R) = G_C P(R) \cap \overline{P}(R)$.

Then it always hold 1. $\Rightarrow$ 2. $\Rightarrow$ 3.

Relations with Auslander and Bass classes

${}_RC_S$, $S = \mathrm{End}_R(C)$.

$\mathcal{A}_C(S)$ consists of all left S-modules M satisfying:

A1) $\mathrm{Tor}^S_{\geq 1}(C, M) = 0$,

A2) $\mathrm{Ext}^{\geq 1}_R(C, C \otimes_S M) = 0$,

A3) the canonical map $\mu_M : M \to \mathrm{Hom}_R(C, C \otimes_S M)$ is an isomorphism of S-modules.

$\mathcal{B}_C(R)$ consists of all left R-modules N satisfying:

B1) $\mathrm{Ext}^{\geq 1}_R(C, N) = 0$,

B2) $\mathrm{Tor}^S_{\geq 1}(C, \mathrm{Hom}_R(C, N)) = 0$,

B3) the canonical map $\nu_N : C \otimes_S \mathrm{Hom}_R(C, N) \to N$ is an isomorphism of R-modules.

Relations with Auslander and Bass classes

If R is commutative and C is semidualizing, Proposition 3.6 in

Y. Geng and N. Ding, *W-Gorenstein modules*, J. Algebra **325** (2011), 132–146.

shows that

$$C-GP(R)=G_CP(R)\cap \mathcal{B}_C(R),$$

where $C-GP(R)$ is the class of syzygies of exact, $Hom_R(C\otimes_R Q,-)$-exact and $Hom_R(-,C\otimes_R Q)$-exact complexes

$$\cdots \to C\otimes_R Q^i \to C\otimes_R Q^{i+1} \to \cdots$$

with $Q, Q^i \in P(R)$.

Definition

A left R-module M is said to be faithful if:
$\mathrm{Hom}_R(M, N) = 0 \Rightarrow N = 0.$
A left R-module M is self-small if
$\mathrm{Hom}_R(M, M^{(I)}) \cong \mathrm{Hom}_R(M, M)^{(I)}$ *for every set I.*

Theorem

If C_S is faithful the following statements are equivalent.
i) ${}_RC$ is Σ-self-orthogonal and self-small.
ii) $C - GP(R) = G_C P(R) \cap \mathcal{B}_C(R)$.
iii) $Add_R(C) \subseteq \mathcal{B}_C(R)$.
iv) $P(S) \subseteq \mathcal{A}_C(S)$.

Through this section C will be a w-tilting module.

Definition

A module M is said to have G$_C$-projective dimension less than or equal to n, $\mathrm{G_C{-}pd}(M) \leq n$, if there is an exact sequence

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$$

with $G_i \in G_C P(R)$ for every $i \in \{0, ..., n\}$. If n is the least nonnegative integer for which such a sequence exists then $\mathrm{G_C{-}pd}(M) = n$, and if there is no such n then $\mathrm{G_C{-}pd}(M) = \infty$.

Proposition

For an R-module M and a positive integer $n \geq 1$, the following assertions are equivalent:

- $\mathrm{G_C{-}pd}_R(M) \leq n$.
- *There is* $0 \to M \to P \to G \to 0$, *where G is G_C-projective and P admits an exact left $Add_R(C)$-resolution of length n (or, equivalently, there exists an exact sequence* $0 \to G_n \to \cdots \to G_0 \to P \to 0$ *with every G_i in $Add_R(C)$).*
- *There is* $0 \to P \to G \to M \to 0$, *where G is G_C-projective and P admits an exact left $Add_R(C)$-resolution of length* $n-1$. *(Thus $P \in G_C P(R)^{\perp}$).*

Corollary

Every module of finite G_C-projective dimension has a special G_C-precover.

As a consequence we see that when $G_C\text{-pd}_R(M)$ is finite, it can be computed using left $G_CP(R)$-resolutions of M.

Corollary

If M is of finite G_C-projective dimension then $\mathrm{G_C{-}pd}(M) \leq n$ if and only if M has an (exact) left $G_CP(R)$-resolution of length $\leq n$.

Theorem

Let M be an R-module of finite G_C-projective dimension and $n \geq 0$ an integer number. The following conditions are equivalent:

1. $\mathrm{G_C{-}pd}_R(M) \leq n$.
2. $\mathrm{Ext}^i_R(M, X) = 0$ *for all* $i > n$ *and all* $X \in Add_R(C)$.
3. *For every exact sequence of R-modules* $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$, *if each G_i is G_C-projective, then so is K_n.*

It is now clear that the G_C-projective dimension of a module can be computed by means of the Ext functors in a similar manner to the Gorenstein projective or the classical projective dimensions.

Corollary

If M is of finite G_C-projective dimension then
$\mathrm{G_C\!-\!pd}_R(M) =$

$$\sup\{i \in \mathbb{N};\ \mathrm{Ext}^i_R(M, X) \neq 0 \ for\ some\ X \in Add_R(C)\}$$

Proposition

Given a family of R-modules $(M_i)_{i\in I}$, we have:

$$\mathrm{G_C\!-\!pd}_R(\oplus_{i\in I} M_i) = \sup\{\mathrm{G_C\!-\!pd}_R(M_i) \,|\, i \in I\}.$$

Definition

A module M is said to have $\mathscr{P}_C$-projective dimension less than or equal to n, $\mathscr{P}_{\mathrm{C}}-\mathrm{pd}(M) \leq n$, if there is an exact sequence

$$0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0$$

with $A_i \in Add_R(C)$ for every $i \in \{0, ..., n\}$. If n is the least nonnegative integer for which such a sequence exists then $\mathscr{P}_{\mathrm{C}}-\mathrm{pd}(M) = n$, and if there is no such n then $\mathscr{P}_{\mathrm{C}}-\mathrm{pd}(M) = \infty$.

Proposition

Let C be any module. If M is G_C-projective and $\mathscr{P}_{\mathrm{C}}-\mathrm{pd}(M) < \infty$ then $M \in Add_R(C)$.

Theorem

Let C be a Σ-self-orthogonal R-module. For an R-module M and an integer number $n \geq 0$ the following assertions are equivalent:

1. $\mathscr{P}_{\mathrm{C}} - \mathrm{pd}(M) \leq n$.
2. *There is an exact left $Add_R(C)$-resolution of M of length n.*
3. *For every left $Add_R(C)$-resolution*

$$\cdots \to Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} M \xrightarrow{f_{-1}} 0$$

of M, $\mathrm{Ker}(f_{n-1}) \in Add_R(C)$ and the left $Add_R(C)$-resolution $0 \to \mathrm{Ker}(f_{n-1}) \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 \to M \to 0$ is exact.

Corollary

If C is a Σ-self-orthogonal R-module, then every R-module of finite $\mathscr{P}_C$-projective dimension has a special $Add_R(C)$-precover.

Corollary

Let C be a w-tilting module and M any R-module. If M is of finite $\mathscr{P}_C$-projective dimension then $M \in G_C P(R)^{\perp} \cap Add_R(C)^{\perp}$.

$\mathscr{P}_C$-projective dimension

Theorem

Let C be a w-tilting R-module. For every R-module M, $\mathrm{G_C{-}pd}(M) \leq \mathscr{P}_\mathrm{C}{-}\mathrm{pd}(M)$, such that the equality $\mathscr{P}_\mathrm{C}{-}\mathrm{pd}(M) = \mathrm{G_C{-}pd}(M)$ holds true whenever M has a finite $\mathscr{P}_C$-projective dimension.

Corollary

Let C be a w-tilting R-module. If M is of finite $\mathscr{P}_C$-projective dimension, then $\mathscr{P}_\mathrm{C}{-}\mathrm{pd}(M) \leq n$ if and only if $\mathrm{Ext}^{>n}(M, X) = 0$ for every $X \in Add_R(C)$.

Definition

An R-module C is said to be faithfully Σ-self-orthogonal if it is a Σ-self-orthogonal module such that, for every left R-module M, if $\mathrm{Hom}_R(C, M) = 0$ then $M = 0$.

Proposition

Every finitely presented w-tilting module C over a commutative ring R is faithfully Σ-self-orthogonal.

Theorem

Let C be a Σ-self-orthogonal R-module. The following assertions are equivalent:

1. *C is faithfully Σ-self-orthogonal.*
2. *If $\phi : I \to M$ is an $Add_R(C)$-precover with $K = \mathrm{Ker}(\phi) \in Add_R(C)^{\perp}$, then ϕ is surjective and $M \in Add_R(C)^{\perp}$.*
3. *If $\cdots \to Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} M \longrightarrow 0$ is a left $Add_R(C)$-resolution of M with $\mathrm{Ker}(f_n) \in Add_R(C)^{\perp}$, then the sequence $0 \to \mathrm{Ker}(f_n) \to \cdots \to Y_1 \to Y_0 \to M \to 0$ is exact.*
4. *Every monic $Add_R(C)$-precover $\phi : I \to M$ of an R-module M is an isomorphism.*
5. *For every R-module M and every $n \in \mathbb{N}$, $\mathscr{P}_{\mathrm{C}}{-}\mathrm{pd}(M) \leq n$ if and only if M admits a left $Add_R(C)$-resolution of length n.*

Corollary

Let C be a faithfully Σ-self-orthogonal R-module.
If $\mathscr{P}_{\mathrm{C}}-\mathrm{pd}(M) \leq n$ then every left $Add_R(C)$-resolution of M is exact.