MORPHISMS DETERMINED BY OBJECTS AND FLAT COVERS

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Motivating problem

**Problem**

- *Fix an additive category, for example a module category.*
- *Is there a procedure for constructing morphisms ending at a fixed object?*

More precisely, we are looking for:

- *Invariants* of morphisms ending at some fixed object.
- *Constructions* for universal morphisms with respect to these invariants.
Our strategy

We combine two concepts:

- Functors/morphisms determined by objects [Auslander, 1978]
- The existence of flat covers [Bican–El Bashir–Enochs, 2001]
Our Setting

We fix:
- $A = \text{an additive category, for example a module category.}$
- $C = \text{a set of objects of } A, \text{ viewed as full subcategory.}$

Definition

- A C-module is an additive functor $C^{\text{op}} \rightarrow \text{Ab}.$
- The category of C-modules is denoted by $(C^{\text{op}}, \text{Ab}).$

Example

$C = \{C\}$ with $\Gamma := \text{End}_A(C).$ Then $(C^{\text{op}}, \text{Ab}) = \text{Mod} \Gamma.$
The pair of categories $A$ and $C$ gives a functor

$$A \longrightarrow (C^{\text{op}}, \text{Ab}), \quad X \mapsto \text{Hom}_A(C, X)$$

by setting

$$\text{Hom}_A(C, X) := \text{Hom}_A(-, X)|_C.$$
For a morphism $\alpha : X \to Y$ in $A$ consider the image

$$\text{Im} \, \text{Hom}_A(C, \alpha) \subseteq \text{Hom}_A(C, Y)$$

of the induced morphism

$$\text{Hom}_A(C, X) \longrightarrow \text{Hom}_A(C, Y) \quad \text{in} \quad (C^{\text{op}}, \text{Ab}).$$
An existence result

**Theorem**

Suppose we have:

- $A$ = a locally finitely presented additive category.
- $C$ = a set of finitely presented objects.
- $Y \in A$ an object and $H \subseteq \text{Hom}_A(C, Y)$ a submodule.

There exists an (essentially unique) morphism $\alpha: X \to Y$ in $A$ such that:

- $\text{Im} \text{Hom}_A(C, \alpha) = H$ and any morphism $\alpha': X' \to Y$ with $\text{Im} \text{Hom}_A(C, \alpha') \subseteq H$ factors through $\alpha$.
- $\alpha$ is right minimal (i.e. any $\phi \in \text{End}_A(X)$ with $\alpha \phi = \alpha$ is invertible).


<table>
<thead>
<tr>
<th>Definition (Auslander)</th>
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<tbody>
<tr>
<td>A morphisms $\alpha: X \to Y$ in $A$ is <strong>right C-determined</strong> if for every morphisms $\alpha': X' \to Y$</td>
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<tr>
<td>$\text{Im} \ Hom_A(C, \alpha') \subseteq \text{Im} \ Hom_A(C, \alpha) \implies \alpha'$ factors through $\alpha$</td>
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For $A = \text{Mod} \Lambda$ and $C = \{C\}$, the theorem is due to Auslander.
**Example: almost split morphisms**

**Definition (Auslander–Reiten)**

A morphism $\alpha: X \to Y$ is right almost split if

- $\alpha$ is not a retraction, and
- $\alpha': X' \to Y$ not a retraction implies $\alpha'$ factors through $\alpha$.

**Proposition (Auslander)**

A morphism $\alpha: X \to Y$ in an additive category is right almost split if and only if

- $\text{End}(Y)$ is a local ring,
- $\alpha$ is right determined by $Y$,
- $\text{Im} \text{Hom}(Y, \alpha) = \text{rad} \text{End}(Y)$.
Existence of almost split split morphisms

**Corollary**

Let $Y$ be a finitely presented object in a locally finitely presented additive category such that $\text{End}(Y)$ is local. Then there exists a (right minimal) right almost split morphism $X \to Y$.

For module categories, this is due to Auslander.
Almost split sequences

In a module category, a right minimal and right almost split morphism \( \alpha : X \to Y \) induces an almost split sequence

\[
0 \to \text{Ker}(\alpha) \to X \to Y \to 0
\]

provided \( Y \) is not projective.

Example

In \( \text{Mod} \mathbb{Z} \) there is no almost split sequence

\[
0 \to A \to B \to \mathbb{Q} \to 0.
\]
Locally finitely presented categories

**Definition (Crawley-Boevey)**

An additive category $A$ is **locally finitely presented** if
- $A$ has direct limits (= filtered colimits),
- the isoclasses of finitely presented objects in $A$ form a set,
- every object is a direct limit of finitely presented objects.

$X \in A$ is **finitely presented** if $\text{Hom}_A(X, -)$ preserves direct limits.

**Example**

Let $\Lambda$ be a ring.
- $\text{Mod} \Lambda = \text{the category of } \Lambda\text{-modules}$
- $\text{Flat} \Lambda = \text{the category of flat } \Lambda\text{-modules}$
When is a morphism determined by objects?

**Proposition**

For a morphism $\alpha : X \to Y$ are equivalent:

- $\alpha$ is determined by a set of finitely presented objects.
- There is a decomposition $X = X' \oplus X''$ such that $\text{Ker}(\alpha|_{X'})$ is pure injective and $\alpha|_{X''} = 0$. 
Functors and morphisms determined by objects were introduced in 1978 by Maurice Auslander in his celebrated Philadelphia notes.

The review begins: *This extremely long paper (244 pages) is devoted to the investigation of functors and morphisms determined by objects.*

The review ends: *The paper is clearly and concisely written. However, in view of the length of the paper, a table of contents would have been very useful.*

Auslander himself was very passionate about this work, but ...
An open problem

Problem (Auslander)

Describe the right $C$-determined morphism $\alpha: X \to Y$ with $\text{Im} \, \text{Hom}_A(C, \alpha) = 0$.

Note: $\alpha = 0$ when $C$ contains a generator of $A$. 
The Auslander bijection

For a category $A$, the morphisms ending at $Y \in A$ are pre-ordered:

$$\alpha' \leq \alpha \iff \alpha' \text{ factors through } \alpha$$

Write $[A/Y]$ for this poset (after identifying $\alpha' = \alpha$ when $\alpha' \leq \alpha$ and $\alpha \leq \alpha'$).

**Theorem (Ringel)**

Let $\Lambda$ be an Artin algebra. For $Y \in \text{mod } \Lambda$ there is an isomorphism

$$\colim_{C \in \text{mod } \Lambda} \text{sub}(\text{Hom}_\Lambda(C, Y)) \overset{\sim}{\longrightarrow} [\text{mod } \Lambda/Y]$$

given by the assignment

$$\text{Hom}_\Lambda(C, Y) \ni H \mapsto \alpha_{C,H}.$$
The proof of the existence result (for C-determined morphisms) is based on:

**Theorem (Bican–El Bashir–Enochs, 2001)**

*Every additive functor admits a flat cover.*

A **flat cover** is the analogue of a projective cover (replacing the term ‘projective’ by ‘flat’).

Next we explain:

**Theorem**

*Flat covers are projective covers.*
For an additive category $A$ we write

$$\text{Fp}(A^{\text{op}}, \text{Ab})$$

for the category of functors $F : A^{\text{op}} \to \text{Ab}$ having a presentation

$$\text{Hom}_A(-, X) \to \text{Hom}_A(-, Y) \to F \to 0.$$

**Proposition**

Let $A$ be locally finitely presented. Then $\text{Fp}(A^{\text{op}}, \text{Ab})$ is an abelian category and

$$A \to \text{Fp}(A^{\text{op}}, \text{Ab}), \quad X \mapsto \text{Hom}_A(-, X)$$

identifies $A$ with the full subcategory of projective objects.
Let \( A \) be a locally finitely presented category.

- For an additive functor \( F : (\text{fp } A)^{\text{op}} \to \text{Ab} \), the unique functor \( \tilde{F} : A^{\text{op}} \to \text{Ab} \) extending \( F \) and preserving filtered colimits in \( A \) is finitely presented and admits a minimal projective presentation in \( \text{Fp}(A^{\text{op}}, \text{Ab}) \).

- The assignment \( F \mapsto \tilde{F} \) provides a fully faithful right adjoint to the functor

\[
\text{Fp}(A^{\text{op}}, \text{Ab}) \longrightarrow ((\text{fp } A)^{\text{op}}, \text{Ab}), \quad F \mapsto F|_{\text{fp } A}.
\]
Example: $A = \text{Flat} \Lambda$

Let $\Lambda$ be a ring and $A = \text{Flat} \Lambda$. Then

$$fp A = \text{proj} \Lambda \quad \text{and} \quad ((fp A)^{\text{op}}, Ab) \xrightarrow{\sim} \text{Mod} \Lambda$$

For $Y \in \text{Mod} \Lambda$ the theorem yields a projective cover

$$\text{Hom}_A(-, X) \longrightarrow \tilde{Y} \quad \text{in} \quad \text{Fp}(A^{\text{op}}, Ab).$$

Evaluation at $\Lambda$ then gives a flat cover

$$X = \text{Hom}_A(\Lambda, X) \longrightarrow \tilde{Y}(\Lambda) = Y.$$
We thank him:

Photo: Gordana Todorov