Generalized Lie derivations on Lie ideals

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Definition

- A ring R is ***prime*** if for $a, b \in R$,$aRb = (0)$ implies that $a = 0$ or $b = 0$, and is ***semiprime*** in case $aRa = (0)$ implies $a = 0$.

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- An additive mapping $D : R \to R$ is called a ***derivation*** if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$.
- An additive map F of a ring R into itself is called a ***generalized derivation*** if there exists a derivation D of R such that $F(xy) = F(x)y + xD(y)$ for all $x, y \in R$.

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Example

Let S be any ring and let $T = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} | x, y \in S \right\}$. Define $F : T \to T$ by $F(x) = (e_{11} + e_{12})x - xe_{12}$. F is a generalized derivation with the associated derivation D such that $D(x) = [e_{12}, x]$.

Definition

Let R be a ring.

- An additive mapping $T : R \to R$ is called a left centralizer (*right centralizer*) if $T(xy) = T(x)y$ $\left(T(xy) = xT(y)\right)$ for all $x, y \in R$.

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- An additive mapping $T : R \to R$ is called a Lie centralizer of R if $T\big([x, y]\big) = \big[T(x), y\big]$ (or $T\big([x, y]\big) = \big[x, T(y)\big]$) for all $x, y \in R$.

Definition

- An additive map F of R acts as a homomorphism on a subset $S \subseteq R$, if $F(xy) = F(x)F(y)$ for all $x, y \in S$; acts as a Lie homomorphism on S if
$$F([x,y]) = [F(x), F(y)]$$
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$$F\big([x,y]\big) = \big[F(x), F(y)\big]$$
for all $x, y \in S$.
- A Lie derivation D of R is an additive map satisfying
$$D\big([x,y]\big) = \big[D(x), y\big] + \big[x, D(y)\big]$$
for all $x, y \in R$.

Definition

Let $F : R \to R$ be an additive map.

- F is called a generalized Lie derivation (in the sense of Hvala) if there exists a linear map $D : R \to R$ such that
$$F\big([x, y]\big) = F(x)y - F(y)x + xD(y) - yD(x)$$
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- F is called a generalized Lie derivation (in the sense of Nakajima) if
$$F([x,y]) = [F(x), y] + [x, D(y)]$$
for all $x, y \in R$ where D is a Lie derivation of R.

- *Let U denote the right Utumi quotient ring of R . Let C be the center of U which is called the* ***extended centroid*** *of R. Note that U is also a prime ring and C is a field.*

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T.K. Lee, 1999

- Let R be a semiprime ring. Then every generalized derivation F on a dense right ideal of R is uniquely extended to U and assumes the form $F(x) = ax + D(x)$ for some $a \in U$ and a derivation D on U. Moreover, a and D are uniquely determined by the generalized derivation F.

H.E.Bell and L.C.Kappe, 1989

Let R be a prime ring and D a derivation of R. If D acts as a homomorphism or anti-homomorphism on a nonzero right ideal of R, then $d = 0$ on R.

A.Asma, N.Rehman, A. Shakir 2003

Let R be a 2-torsion free prime ring and L a noncentral Lie ideal of R such that $u^2 \in L$ for all $u \in L$ and D acts as a homomorphism or anti-homomorphism on L, then $D = 0$.

Y. Wang, H. You 2006

Let R be a 2-torsion free prime ring and L a nonzero Lie ideal of R. If D is a derivation of R which acts as a homomorphism or anti-homomorphism on L, then either $D = 0$ or $L \subseteq Z(R)$.

P.B. Liao, C. K. Liu 2009

Let A be a prime algebra with $charA \neq 2$ and extended centroid C. Let R be a noncentral Lie dieal of A and B be a subalgebra of A generated by R and let $F : R \to A$ be a generalized Lie derivation of R. Suppose that A does not satisfy the standard identitiy of degree 18. Then there exist a generalized derivation $G : B \to AC + C$ and a linear map $\tau : R \to C$ such that $F(x) = G(x) + \tau(x)$ for all $x \in R$ and $\tau\big([R, R]\big) = 0$.

M. Ashraf, N. Rehman, S. Ali, R. Mozumder 2010

Let R be a 2-torsion free semiprime ring and I be a nonzero ideal of R. Suppose that R admits a generalized derivation F with associated nonzero derivation D. Further, if

$$F([x,y]) = [D(x), F(y)]$$

for all $x, y \in I$ or

$$F([x,y]) + [D(x), F(y)] = 0$$

for all $x, y \in I$, then R contains a nonzero central ideal.

E. Albas, 2011

Let R be a prime ring with $charR \neq 2$ and F a generalized derivation of R. If

$$F([x,y]) = [F(x), F(y)]$$

for all $x, y \in R$ then either R is commutative or $F = 0$ or $F = I$.

G. Scudo, 2012

Let R be a prime ring with $charR \neq 2$ and C the extended centroid of R, $f(x_1, ...x_n)$ a non-central polynomial over C in n noncommuting variables and F a nonzero generalized derivation of R. The set of all evaluations of $f(x_1, ..., x_n)$ in R is denoted by $f(R)$. If

$$F\big([u, v]\big) = \big[F(u), F(v)\big]$$

for all $u, v \in f(R)$ then $F(x) = x$ for all $x \in R$.

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- $Z(R)$: the center of R
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- L: a noncommutative Lie ideal of R
- D: a derivation of R
- F: a generalized derivation of R associated with the derivation D.
- δ: a derivation of R.

For all $x, y \in L$;

1. $F\big([x,y]\big) = \big[D(x), F(y)\big]$
2. $F\big([x,y]\big) = \big[F(x), y\big] + \big[x, D(y)\big]$
3. $F\big([x,y]\big) = \big[F(x), y\big] + \big[x, \delta(y)\big]$

Theorem1

Let R be a prime ring with $charR \neq 2$ and L a noncommutative Lie ideal of R. Let F be a nonzero generalized derivation of R associated with the derivation D. Suppose that

$$F\big([x,y]\big) = \big[D(x), F(y)\big]$$

for all $x, y \in L$. Then $F = 0$ on $x \in R$.

Lemma 1

Let R be a prime ring with $CharR \neq 2$ and L be a noncommutative Lie ideal of R, and $b, c \in R$. If

$$c[x,y] - [x,y]b = \left[[b,x], cy - yb\right]$$

for all $x, y \in L$ then $b = c \in Z(R)$

Theorem 2

Let R be a prime ring with $charR \neq 2$ and L a noncommutative Lie ideal of R. Let F be a generalized derivation of R associated with the derivation D. Suppose that

$$F\big([x,y]\big) = \big[F(x), y\big] + \big[x, D(y)\big]$$

for all $x, y \in L$. Then $F(x) = \lambda x + D(x)$ for all $x \in R$ and for some $\lambda \in C$.

Theorem 3

Let R be a prime ring with $charR \neq 2$ and L a noncommutative Lie ideal of R. Let F be a generalized derivation of R associated with the derivation D and δ be any derivation of R. Suppose that

$$F\left([x,y]\right) = \left[F(x),y\right] + \left[x,\delta(y)\right]$$

for all $x,y \in L$ then, $\delta = D$ on R and $F(x) = \mu x + D(x)$ for all $x \in R$ and for some $\mu \in C$.

Observations

Let D and F be additive maps of the ring R.

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Example

Let $M_n(\mathbb{F})$ be the $n \times n$ matrix ring over the field $\mathbb{F}$ and $tr : M_n(\mathbb{F}) \to \mathbb{F}$ be the trace mapping. Define $D : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ as $D(x) = [a, x] + tr(x).I$. The additive mapping D is a Lie derivation but not a derivation.

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Example

Let R be a ring and $T : R \to R$ be an additive map. Define $T(x) = ax + \lambda(x)$ where $a \in U$ and $\lambda : R \to C$ an additive mapping such that $\lambda\big([R, R]\big) = 0$. T is a Lie centralizer but not a centralizer.

- Let $1_R \in R$.
  F is a generalized derivation of R.

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- F is a generalized derivation of R.

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  F is a generalized Lie derivation (in the sense of Nakajima) of R.

THANK YOU!

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