Representation embeddings preserve complexity

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(Suppose that R and S are finite-dimensional algebras.)

A **representation embedding** from $\mathrm{mod}\text{-}S$ to $\mathrm{mod}\text{-}R$ is a functor $F : \mathrm{mod}\text{-}S \to \mathrm{mod}\text{-}R$ which is exact, preserves indecomposability and reflects isomorphism. Such a functor has the form $M \mapsto M \otimes_S B_R$ for an (S, R)-bimodule B which is finitely generated and projective as a left S-module.

In what sense(s) does such a functor "preserve the complexity of the category of S-modules"? For example, it should be non-decreasing on dimensions which are some measure of this complexity (Krull-Gabriel dimension = m-dimension; uniserial dimension = breadth/width).

We can also ask whether the original category $\mathrm{mod}\text{-}S$ can be recovered from its image in $\mathrm{mod}\text{-}R$.

Theorem

Suppose that R and S are finite-dimensional algebras and that there is a representation embedding from $\mathrm{mod}\text{-}S$ *to* $\mathrm{mod}\text{-}R$*. Then* $\mathrm{KG}(S) \le \mathrm{KG}(R)$ *($\mathrm{KG} =$ Krull-Gabriel dimension, equivalently m-dimension). Similarly for uniserial dimension=breadth.*

Corollary

If R is a finite-dimensional algebra of wild representation type then the width of the lattice of pp formulas for R-modules is ∞. If R also is countable then there is a superdecomposable pure-injective R-module.

The dimensions appearing above can be defined either by successive localisations of the category, $(\mathrm{mod}\text{-}R, \mathbf{Ab})^{\mathrm{fp}}$, of finitely presented (additive) functors on finitely presented modules or by successive collapsings on the lattice of pp conditions for R-modules.

Pp conditions are, in turn, equivalent to pointed finitely presented modules.

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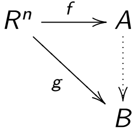
Pp conditions in n free variables are equivalent to n-pointed finitely presented modules, that is, pairs $(A, \overline{a})$ where A is finitely presented and $\overline{a}$ is an n-tuple of elements of A, equivalently, morphisms $R^n \to A$ with A finitely presented.

Morphisms $R^n \to A$ (with A finitely presented) are naturally pre-ordered by $(f : R^n \to A) \geq (g : R^n \to B)$ iff g factors initially through f.



(If R is a finite-dimensional algebra then each equivalence class has a minimum representative. From the model-theoretic viewpoint this is a (minimal) **free realisation**, $(C_\varphi, \overline{c}_\varphi)$, of the corresponding pp condition φ.)

Denote by pp_R^n the resulting partially ordered set. This is a lattice, with join being given by direct sum and meet by pushout.

Theorem

If R, S are finite-dimensional algebras and there is a representation embedding from $\mathrm{mod}\text{-}S$ to $\mathrm{mod}\text{-}R$, then there is an embedding of lattices from pp_S^1 to pp_R^n for some n.

To get this embedding: choose a finite (n)-tuple $\overline{t}$ from B which generates the module B_R (where B is the bimodule such that $- \otimes B$ is the representation embedding). Given an element (C, c) of pp_S^1 the corresponding element of pp_R^n is the n-pointed module $(C \otimes B, c \otimes \overline{t})$.

Theorem (Lorna Gregory)

Suppose that R and S are finite-dimensional algebras and that there is a finitely controlled representation embedding $F : \mathrm{mod}\text{-}S \to \mathrm{mod}\text{-}R$. Then there is an interpretation functor from the definable category generated by the image of F to $\mathrm{Mod}\text{-}S$, with image all of $\mathrm{Mod}\text{-}S$.

The condition that F be (finitely) controlled is that there is a subcategory $\mathcal{C}$ (the additive closure of finitely many indecomposables) of $\mathrm{mod}\text{-}R$ such that for every $A, B \in \mathrm{mod}\text{-}S$ we have $(FA, FB) = F(A, B) \oplus (FA, FB)_{\mathcal{C}}$, where the latter consists of the morphisms from FA to FB which factor through $\mathcal{C}$.

Corollary

Suppose that R is a finitely controlled-wild finite-dimensional algebra. Then the theory of R-modules interprets the word problem for (semi)groups, in particular it is undecidable.

An **interpretation functor** between definable categories is one which commutes with direct products and direct limits (equivalently it is an (additive) interpretation in the model-theoretic sense).

Theorem (Krause for $\mathcal{C}$ locally finitely presented, Prest in general)

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are definable categories. Then the interpretation functors $I : \mathcal{C} \to \mathcal{D}$ are in natural (2-categorical) bijection with the exact functors $\mathrm{fun}(\mathcal{D}) \to \mathrm{fun}(\mathcal{C})$ (alternatively written $\mathbb{L}^{\mathrm{eq}+}(\mathcal{D}) \to \mathbb{L}^{\mathrm{eq}+}(\mathcal{C})$).

$\mathrm{fun}(\mathcal{C})$ is, in the case $\mathcal{C} = \mathrm{Mod}\text{-}R$, the skeletally small abelian category $(\mathrm{mod}\text{-}R, \mathbf{Ab})^{\mathrm{fp}}$ and in general is a quotient of such a functor category; $\mathbb{L}^{\mathrm{eq}+}(\mathcal{C})$ denotes the category of pp-defined sorts and pp-defined functions on objects of $\mathcal{C}$. The lattice pp_R^1 is contained in, and in some sense generates, $\mathbb{L}^{\mathrm{eq}+}(\mathrm{Mod}\text{-}R)$ (it is the lattice of finitely generated subfunctors of the forgetful functor).