Leavitt Path Algebras with at most countably many irreducible representations

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Directed Graphs

A **directed graph** $E = (E^0, E^1, r, s)$ consists of a set E^0 of **vertices**, a set E^1 of **edges** and maps r, s from E^1 to E^0. For each $e \in E^1$, say,

$$\overset{u}{\bullet} \;\underset{e^*}{\overset{e}{\rightleftarrows}}\; \overset{v}{\bullet}$$

, $s(e) = u$ is called the **source** of e and $r(e) = v$ the **range** of e and e^* is called the ghost edge with $s(e^*) = v$ and $r(e^*) = u$. A **finite path** α of length $n > 0$ is a finite sequence of edges $\alpha = e_1 e_2 \cdots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i = 1, \cdots, n-1$. In this case $\alpha^* = e_n^* \cdots e_2^* e_1^*$.
A vertex u is called a **sink** if it emits no edges. If u is not a sink and emits finitely many edges, we say u is a **regular vertex**. If u emits infinitely many edges, we say u is an **infinite emitter.**

$$\begin{array}{ccccc}
\overset{v_1}{\bullet} & \overset{e_1}{\longrightarrow} & \overset{v_2}{\bullet} & \overset{e_2}{\longrightarrow} & \overset{v_3}{\bullet} \\
 & & \downarrow f & \searrow g & \downarrow e_3 \\
 & & \overset{v_5}{\bullet} & \overset{\infty}{\longrightarrow} & \overset{v_4}{\bullet}
\end{array}$$

In the above diagram, v_5 is an infinite emitter; v_4 is a sink; v_1, v_2, v_3 are regular vertices.

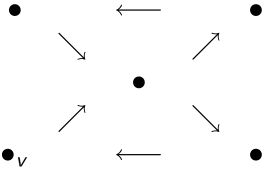
Leavitt path algebras

Let E be a directed graph and K be any field. The **Leavitt path algebra** $L_K(E)$ of the graph E with coefficients in K is the K-algerbra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents together with a set of variable $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:
(1) $s(e)e = e = er(e)$ for all $e \in E^1$.
(2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.
(3) (The "CK-1 relations") For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$.
(4) (The "CK-2 relations") For every regular vertex $v \in E^0$,

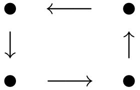
$$v = \sum_{e \in E^1, s(e) = v} ee^*.$$

Some Definitions

An **closed path** is a finite sequence of edges $c = e_1 e_2 \cdots e_n$ where $r(e_n) = s(e_1)$. In this case, $v = s(e_1) = r(e_n)$ is called the **base** of the closed path c.



; An example of a **cycle** is:



Theorem

(X.W. Chen): The set of infinite paths tail-equivalent an infinite path p in E can be made a simple module S_p over $L_K(E)$ and if q is an infinite path NOT tail-equivalent to p, then $S_q \ncong S_p$.

Some Results

If L has at most countably many non-isomorhic irreducible representations, we say L is **CIRT**.

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- **Definition**: Suppose no two cycles in the graph E have a common vertex. Given two cycles c and c', define $c \geq c'$ if there is a path from a vertex on c to a vertex on c'. Because distinct cycles have no common vertex, it is clear that the relation $\geq$ is anti-symmetric and hence is a partial order on the set of distinct cycles in E (where two cycles g, h are considerd distinct if $g^0 \neq h^0$)

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- The graph E contains cycles without exits.

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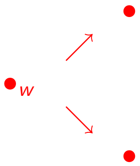

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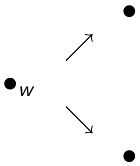

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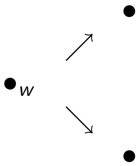

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- **Proposition 4**: (i) A vertex v is a line point $\Longleftrightarrow$ Lv (vL) is a simple left (right) ideal of L.
- (ii) The ideal generated by all the line points in E is the $Soc(L)$ of L.

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$$0 < I_1 < \cdots < I_\alpha < I_{\alpha+1} < \cdots \qquad (\alpha < \tau) \qquad (**)$$

where, τ is a countable ordinal, for each $0 \leq \alpha < \tau$, $I_{\alpha+1}/I_\alpha$ is a direct sum of at most countably many matrix rings over K and/or over $K[x, x^{-1}]$. Moreover, K will be a countable field whenever E contains cycles.

When the graph has no cycles.

In the above Theorem, if the graph E is acyclic, then for every $\alpha < \tau$, $I_{\alpha+1}/I_{\alpha}$ is isomorphic to a direct sum of matrix rings over K and hence is a direct sum of simple modules. Thus the chain $(**)$ becomes the socular chain for L, namely,

$$0 < I_1 < \cdots < I_{\alpha} < I_{\alpha+1} < \cdots \qquad (\alpha < \tau) \qquad (**)$$

where, for each $\alpha < \tau$, $I_{\alpha+1}/I_{\alpha} \cong Soc(L/I_{\alpha})$. Note that, in this case, there is no restriction on the cardinality of the field K. Also L becomes semi-artinian (that is, every non-zero left (right) R-module contains a simple submodule) and von-Neumann regular.

Finitely many non-isomorphic irreducible representations

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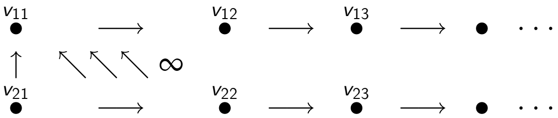
- **Question**: Which finite distributive lattice can occur as the lattice of all ideals in the Leavitt path algebra $L_K(E)$ of finite irreducible representation type ?
- **Answer:** George Bergman proved that every finite distributive lattice can be represented as the lattice of ideals of a unital ultramatricial algebra. On the other hand, Raeburn showed that this unital ultramatricial algebra is Morita-equivalent to a Leavitt path algebra $L_K(E)$ of an acyclic graph. For this graph E, the conditions in Theorem (ii) necessarily hold. Consequently, every distributive lattice can be realized as the lattice of all ideals of a Leavitt path algebra of finite irreducible representation type.

Existence of isomorphism classes of simples

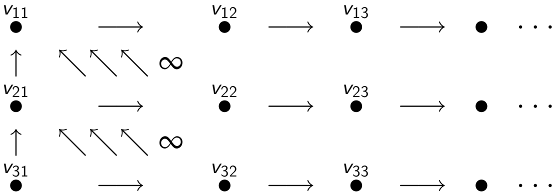
1. Let P_1 be the graph $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots\cdots$. Then $L_K(P_1)$ is a direct sum of isomorphic simple left/right ideals and thus all simple left/right modules over $L_K(P_1)$ are isomorphic.
2. Let P_2 be the graph



where $\nwarrow\nwarrow\nwarrow \infty$ denotes that each of the infinitely many vertices v_{2n} $(n \geq 2)$ is connected to the vertex v_{11} by an edge. Now the line points in the graph P_2 are the vertices $v_{11}, v_{12}, v_{13}, \cdots$ and they generate the socle S of P_2 which is a direct sum of isomorphic (faithful) simple left/right modules. Also, $P_2/S \cong L_K(P_1)$ is a direct sum of isomorphic simple modules annihilated by the ideal S. Thus $L_K(P_2)$ has exactly two non-isomorphic simple modules.

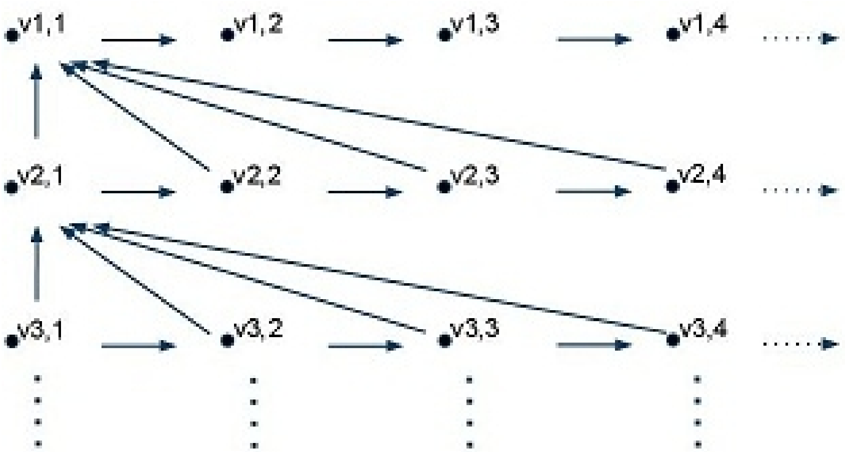
Exactly three isomorphism classes of simples

3. Let P_3 be the graph



isomorphism classes of simple modules

Let $P_\omega = \bigcup_{n\in\mathbb{N}} P_n$ be the "Pyramid" graph of length ω constructed inductively and represented pictorially as follows.



Exactly Omega+1 isomorphism classes of simple modules

The graph $P_{\omega+1}$ is obtained from the graph P_ω by adding a single vertex $v_{\omega+1}$ and connecting it by an edge to each of the vertices v_{j1} for $j < \omega$ in the graph P_ω. Specifically, $(P_{\omega+1})^0 = (P_\omega)^0 \cup \{v_{\omega+1}\}$, $(P_{\omega+1})^1 = (P_\omega)^1 \cup \{e_{\omega+1,j} : j < \omega\}$ where, for each j, $s(e_{\omega+1,j}) = v_{\omega+1}$ and $r(\ e_{\omega+1,j}) = v_{j1}$. Clearly, $I_\omega =< P_\omega >\cong L_K(P_\omega)$ and $L_K(P_{\omega+1})/I_\omega \cong K$ and so $L_K(P_{\omega+1})$ will have $\omega + 1$ distinct isomorphism classes of simple $L_K(P_{\omega+1})$-modules.

For cardinal κ (finite or infinite), there exists a Leavitt path algebra having exactly κ distinct isomorphism classes of simple left/right modules.