Approximations and Mittag-Leffler modules

Algebraic structures and their applications, Spineto
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Preliminaries

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- we work in ZFC

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- characterize cotorsion pairs $(\mathcal{A}, \mathcal{B})$ with $\mathcal{B} = \varinjlim \mathcal{B}$
- byproduct: $\mathcal{A}$ is covering $\implies \mathcal{A} = \varinjlim \mathcal{A}$ (an instance of Enochs' problem)

Module approximation tools

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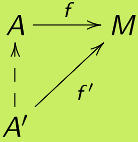
Definition

A class of modules $\mathcal{A}$ is precovering if for each module M there is $f \in \mathrm{Hom}_R(A, M)$ with $A \in \mathcal{A}$ such that each $f' \in \mathrm{Hom}_R(A', M)$ with $A' \in \mathcal{A}$ factorizes through f:



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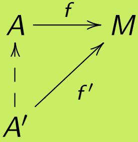

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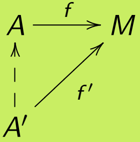

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If $\mathcal{A}$ provides for covers for all modules, then $\mathcal{A}$ is called a covering class.

Transfinite extensions

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Let $\mathcal{A} \subseteq$ Mod-R. A module M is $\mathcal{A}$-filtered (or a transfinite extension of the modules in $\mathcal{A}$), provided that there exists an increasing sequence $(M_\alpha \mid \alpha \leq \sigma)$ consisting of submodules of M such that $M_0 = 0$, $M_\sigma = M$,

- $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$, and
- for each $\alpha < \sigma$, $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{A}$.

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The class ${}^\perp\mathcal{C} = \text{KerExt}^1_R(-, \mathcal{C})$ is filtration closed for each class of modules $\mathcal{C}$.

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In particular, so are the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ of all modules of projective and flat dimension $\leq n$, for each $n < \omega$.

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A class of modules $\mathcal{A}$ is deconstructible, provided there is a cardinal κ such that $\mathcal{A} = \mathsf{Filt}(\mathcal{A}^{<\kappa})$ where $\mathcal{A}^{<\kappa}$ denotes the class of all $< \kappa$-presented modules from $\mathcal{A}$.

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Šťovíček

Every deconstructible class is precovering.

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$\text{Filt}(\mathcal{A}) = \mathcal{A}$, and $\mathcal{B}$ is closed under $\prod$.

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Eklof, Trlifaj; Šťovíček

If $\mathfrak{C}$ is generated by a *set*, then $\mathcal{A}$ is deconstructible.

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Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair with $\mathcal{B} = \varinjlim \mathcal{B}$. Then all modules in $\mathcal{A}$ are $\mathcal{A}^{<\aleph_1}$-filtered. Consequently, $\mathfrak{C}$ is generated by (a representative subset of) $\mathcal{A}^{<\aleph_1}$, and $\mathcal{B}$ is a definable class.

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$$(\forall i \in I)(\exists j_i \geq i)(\forall k > j_i)\ \mathrm{Im}(\mathrm{Hom}_R(f_{j_i i}, C)) = \mathrm{Im}(\mathrm{Hom}_R(f_{ki}, C)).$$

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The concept of a $\mathcal{C}$-stationary module is not a new one. Under the name *relative Mittag-Leffler*, these modules have been studied from the 70s (Raynaud, Gruson), through 90s (Rothmaler, Zimmermann) to the recent times (Angeleri Hügel, Herbera).

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8. $(\varinjlim \mathcal{A})^{<\aleph_1}$ *consists of* $\mathcal{B}$*-stationary modules;*
9. *Every pure-epimorphic image of a module from* $\mathcal{A}$ *is* $\mathcal{B}$*-stationary.*

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5. K *is Σ-pure-split.*

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A module M is Mittag-Leffler if, for each system of left R-modules $(N_i \mid i \in I)$, the canonical map $M \otimes_R \prod_{i \in I} N_i \to \prod_{i \in I} M \otimes_R N_i$ is monic.

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Remark (comparison with $\mathcal{C}$-stationarity)

A module M is $\mathcal{C}$-stationary, iff the canonical map
$M \otimes_R (C^*)^I \to (M \otimes_R C^*)^I$ is monic for all $C \in \mathcal{C}$ and all sets I.

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- The result originally proved by Š. and Trlifaj for countable rings, assuming SCH.
- Later, Bazzoni and Šťovíček found a proof in ZFC.

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This gives plenty of new examples of non-precovering classes closed under filtrations and pure submodules, e.g. over hereditary Artin algebras of infinite representation type.

References

1. L. Angeleri Hügel, D. Herbera: "Mittag-Leffler conditions on modules", Indiana Univ. Math. J. 57 (2008), 2459–2518.

2. L. Angeleri Hügel, J. Šaroch, J. Trlifaj: "Approximations and Mittag-Leffler conditions", preprint.

3. A. Slávik, J. Trlifaj: "Approximations and locally free modules", Bull. London Math. Soc. 46 (2014), 76–90.

4. J. Šaroch, J. Šťovíček: "The countable Telescope Conjecture for module categories", Adv. Math. 219 (2008), 1002–1036.