Generalized injectivity and approximations

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Basic Notions and Definitions

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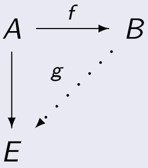
- An R module E is *injective* if, for every module B and every submodule A of B, every monomorphism $f : A \hookrightarrow B$ can be extended to map $g : B \to E$, that is the following diagram commutes



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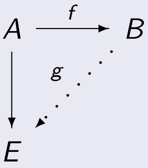

- If an injective R module E is an maximal essential extension of an R module M, then E is said to be an *injective envelope* of M.

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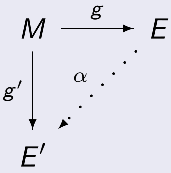

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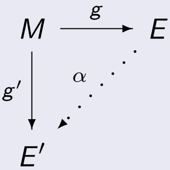

  with $E' \in \mathcal{C}$ can be completed by $\alpha : E \to E'$ to a commutative diagram.
- If moreover the diagram can be completed only by an automorphism α, we call g a $\mathcal{C}$*-envelope* (or a *minimal left* $\mathcal{C}$*-approximation*) of M.

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- Dually, one can define the notions of a *$\mathcal{C}$-precover* (= *right $\mathcal{C}$-approximation*) and a *$\mathcal{C}$-cover* (= a *minimal right $\mathcal{C}$-approximation*) of a module M, and of a *(pre)covering* class of modules.

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- A $\mathcal{C}$-preenvelope $f : M \to C$ of M is called *special*, provided that f is injective and Coker$f \in {}^{\perp}\mathcal{C}$.

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- A module Q is called *quasi-injective* if it is Q-injective.

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- M is a $C3$-*module* in case the following holds true: if A and B are direct summands in M and $A \cap B = 0$, then $A + B$ is a direct summand in M.

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$\mathcal{C}_i$:= the class of all C_i-modules for $i = 1, 2, 3$.
$\mathcal{C}_4$:= classes of all quasi-continuous modules,
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$$\mathcal{C}_2 \subseteq \mathcal{C}_3 \quad \text{and} \quad \mathcal{C}_6 \subseteq \mathcal{C}_5 = \mathcal{C}_1 \cap \mathcal{C}_2 \subseteq \mathcal{C}_4 = \mathcal{C}_1 \cap \mathcal{C}_3 \subseteq \mathcal{C}_3.$$

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- Every R-module has an injective envelope which is unique up to isomorphism.(see Enochs, E.E. and Jenda, O.M.G.: *Relative Homological Algebra*, Chapter 3)
- Let $\mathcal{C}$ be a class of pure-injective modules, such that $\mathcal{C}$ is closed under direct summands. Let $f \in \mathrm{Hom}_R(M, C)$ be a $\mathcal{C}$-preenvelope of M. Then there is a decomposition $C = D \oplus E$, such that Img$f \subseteq D$ and $f : M \to D$ is left minimal. In particular, $f : M \to D$ is a $\mathcal{C}$-envelope of M. (see H. Krause,M. Saorin, On minimal approximations of modules, Contemp. Math. 229 (1998), 227236.)

- Let R be a ring. Then every module has a special fp-injective preenvelope.(see Göbel R., Trlifaj J., *Approximations and Endomorphism Algebras of Modules*, Chapter 6.)

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- Every module M has a minimal quasi-injective extension, which is unique up to isomorphism.(The quasi-injective hull is not the same as quasi-injective preenvelope) (see Mohamed S.H., Müller B.J. *Continous and Discrete Modules*, Chapter 1.)

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If these conditions are satisfied, then R is a right noetherian right V-ring; moreover, all semisimple modules are injective.

Example

Let R be a hereditary two-sided noetherian right V-ring. Then the classes of all quasi-injective and all injective modules coincide by [Proposition 5.19(3)]. [Cozzens, J., Faith, C.: *Simple Noetherian Rings*, Cambridge Univ. Press, Cambridge 1975].

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Open Question

How about the structure of rings that satisfy the equivalent conditions of Theorem 1 for $i = 2, 5$?

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If these conditions are satisfied, then R is right artinian.

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- (3) $\mathcal{C}_1 = \text{Mod-}R$,

Open Problem

The results, for the class of C1 modules, are proved for the domain case and the noetherian setting. Can one extend that results to arbitrary rings?

THANK YOU FOR YOUR ATTENTION