Generalized injectivity and approximations

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(joint work with Jan TRLIFAJ)

Charles University in Prague
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An $R$-module $E$ is injective if, for every module $B$ and every submodule $A$ of $B$, every monomorphism $f : A \to B$ can be extended to map $g : B \to E$, that is the following diagram commutes:

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
E & & E
\end{array}
\]

If an injective $R$-module $E$ is an maximal essential extension of an $R$-module $M$, then $E$ is said to be an injective envelope of $M$.
An $R$ module $E$ is injective if, for every module $B$ and every submodule $A$ of $B$, every monomorphism $f : A \hookrightarrow B$ can be extended to map $g : B \rightarrow E$, that is the following diagram commutes:

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\begin{array}{ccc}
A & \xrightarrow{f} & B \\
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\]
Definitions

- An \( R \) module \( E \) is *injective* if, for every module \( B \) and every submodule \( A \) of \( B \), every monomorphism \( f : A \hookrightarrow B \) can be extended to map \( g : B \to E \), that is the following diagram commutes.

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- If an injective \( R \) module \( E \) is an maximal essential extension of an \( R \) module \( M \), then \( E \) is said to be an *injective envelope* of \( M \).
Definitions

A homomorphism \( g : M \rightarrow E \) is a \( C \)-preenvelope (or a left \( C \)-approximation) of a module \( M \), provided that \( E \in C \) and each diagram

\[
\begin{array}{ccc}
M & \xrightarrow{g} & E \\
\downarrow & & \downarrow \\
E' & & \end{array}
\]

can be completed by \( \alpha : E \rightarrow E' \) to a commutative diagram.

If moreover the diagram can be completed only by an automorphism \( \alpha \), we call \( g \) a \( C \)-envelope (or a minimal left \( C \)-approximation) of \( M \).

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Definitions

• A homomorphism \( g : M \to E \) is a \( \mathcal{C} \)-preenvelope (or a left \( \mathcal{C} \)-approximation) of a module \( M \), provided that \( E \in \mathcal{C} \) and each diagram

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Definitions

A class $C \subseteq \text{Mod-}R$ is a preenveloping class, (enveloping class) provided that each module has a $C$-preenvelope ($C$-envelope). Dually, one can define the notions of a $C$-precover (= right $C$-approximation) and a $C$-cover (= a minimal right $C$-approximation) of a module $M$, and of a (pre)covering class of modules. A submodule $A$ of a module $B$ is a pure submodule, ($A \subseteq \ast B$ for short) if for each finitely presented module $F$, the functor $\text{Hom}_R(F, -)$ preserves exactness of the short exact sequence $0 \to A \to B \to B/A \to 0$. A $C$-preenvelope $f : M \to C$ of $M$ is called special, provided that $f$ is injective and $\text{Coker} f \in \perp C$. 

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- A submodule $A$ of a module $B$ is pure submodule, ($A \subseteq^* B$ for short) if for each finitely presented module $F$, the functor $\text{Hom}_R(F, -)$ preserves exactness of the short exact sequence $0 \to A \to B \to B/A \to 0$. 

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Definitions

Modules that are injective with respect to pure embeddings are called pure-injective.

If $M(\lambda)$ is pure-injective for all cardinals $\lambda$, then $M$ is called $\Sigma$-pure-injective.

$M$ is fp-injective, provided that $\text{Ext}^1_R(F, M) = 0$ for each finitely presented left $R$-module $F$.

An $R$-module $N$ is $A$-injective if, for every submodule $X$ of $A$ and any morphism $f: X \to A$ can be extended to map $g: A \to N$.

A module $Q$ is called quasi-injective if it is $Q$-injective.
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Definitions

Let $R$ be a ring and $M$ a module. Then $M$ is a $C_1$-module provided that every submodule of $M$ is essential in a direct summand of $M$, a $C_1$-module can sometimes be called as CS or extending modules; $M$ is a $C_2$-module provided that $A$ is a direct summand in $M$ whenever $A$ is a submodule of $M$ such that $A$ is isomorphic to a direct summand in $M$; $M$ is a $C_3$-module in case the following holds true: if $A$ and $B$ are direct summands in $M$ and $A \cap B = 0$, then $A + B$ is a direct summand in $M$. 

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Let \( R \) be a ring and \( M \) a module. Then

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Definition

A module $M$ is continuous, if $M$ is both $C_1$ and $C_2$; $M$ is quasi-continuous if $M$ is both $C_1$ and $C_3$.

The following implications hold:

$\text{Injective} \implies \text{quasi-injective} \implies \text{continuous} \implies \text{quasi-continuous} \implies C_1.$

Notations

$C_i := \text{the class of all } C_i\text{-modules for } i = 1, 2, 3.$

$C_4 := \text{classes of all quasi-continuous modules},$

$C_5 := \text{classes of all continuous modules},$

$C_6 := \text{the classes of all quasi-injective modules}.$

Thus, we have $C_2 \subseteq C_3$ and $C_6 \subseteq C_5 = C_1 \cap C_2 \subseteq C_4 = C_1 \cap C_3 \subseteq C_3.$

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Approximations of (Generalized) Injective Modules
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- Every $R$-module has an injective envelope which is unique up to isomorphism. (see Enochs, E.E. and Jenda, O.M.G.: *Relative Homological Algebra*, Chapter 3)

- Let $C$ be a class of pure-injective modules, such that $C$ is closed under direct summands. Let $f \in \text{Hom}_R(M, C)$ be a $C$-preenvelope of $M$. Then there is a decomposition $C = D \oplus E$, such that $\text{Img} f \subseteq D$ and $f : M \to D$ is left minimal. In particular, $f : M \to D$ is a $C$-envelope of $M$. (see H. Krause, M. Saorin, On minimal approximations of modules, Contemp. Math. 229 (1998), 227236.)
Injective Modules and Their Generalizations
New Results on Generalized Injective Modules

Let $R$ be a ring. Then every module has a special fp-injective preenvelope. (see G¨obel R., Trlifaj J., Approximations and Endomorphism Algebras of Modules, Chapter 6.)

Every module $M$ has a minimal quasi-injective extension, which is unique up to isomorphism. (The quasi-injective hull is not the same as quasi-injective preenvelope) (see Mohamed S.H., M¨uller B.J. Continuous and Discrete Modules, Chapter 1.)
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Theorem 1

Let $R$ be a ring and $1 < i \leq 6$. Then the following conditions are equivalent:

1. The class $C_i$ is closed under finite direct sums.
2. $C_i$ coincides with the class of all injective modules.
3. $C_i$ is (pre)enveloping.
4. $C_i$ is (pre)covering.

If these conditions are satisfied, then $R$ is a right noetherian right $V$-ring; moreover, all semisimple modules are injective.
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Example


Note

Clearly, each semisimple ring $R$ satisfies the condition of the Theorem 1 for all $1 \leq i \leq 6$.

Corollary

Let $R$ be a ring and $i = 3$ or $i = 4$. Then the equivalent conditions of the Theorem 1 are satisfied, if and only if $R$ is a semisimple ring.

Open Question

How about the structure of rings that satisfy the equivalent conditions of Theorem 1 for $i = 2, 5$?
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Let $R$ be a ring such that either $R$ is right noetherian or $\text{Soc}(R) = 0$. Then the following conditions are equivalent:

1. $C_1$ is (pre)enveloping.
2. $C_1$ consists of $\sum$-pure-injective modules and it is closed under direct products.

If these conditions are satisfied, then $R$ is right artinian.
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Theorem 3

Let $R$ be a commutative noetherian ring, or a commutative domain. Then the following conditions are equivalent:

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2. $R$ is an artinian serial ring with $J^2 = 0$.
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3. \( C_1 = \text{Mod-}R \),
Open Problem

The results, for the class of C1 modules, are proved for the domain case and the noetherian setting. Can one extend that results to arbitrary rings?
THANK YOU FOR YOUR ATTENTION