Cotilting modules over commutative noetherian rings

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Tilting modules

Let R be an associative ring with unit and $n < \omega$.
A (right R–) module T is *n–tilting* provided that

(T1) $\mathrm{pd}_R(T) \leq n$,

(T2) $\mathrm{Ext}_R^k(T, T^{(\kappa)}) = 0$ for all $k \geq 1$ and all κ,

(T3) There is an exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$ with $T_i \in \mathrm{Add}\, T$ for all $i \leq n$.

The *n-tilting class* induced by T is
$T^\perp = \{M \in \text{Mod-}R \mid \mathrm{Ext}_R^k(T, M) = 0 \text{ for all } k \geq 1\}$.

The tilting modules T and T' are *equivalent* if $T^\perp = (T')^\perp$.

Large tilting modules versus finite type

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If R is commutative, then all non-trivial tilting modules are large:

Lemma

Let R be a commutative ring.

- *If* $0 \neq T \in \mathrm{mod}\text{–}R$ *has projective dimension n, then* $Ext_R^n(T, T) \neq 0$.
- **All finitely generated tilting modules are projective.**

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- **All finitely generated tilting modules are projective.**

Theorem (finite type of tilting classes)

Let R be a ring, $n < \omega$, and $\mathcal{T}$ be a class of modules.

- *Then $\mathcal{T}$ is n–tilting, iff there is a set $\mathcal{S} \subseteq \text{mod}-R$ consisting of modules of projective dimension $\leq n$ such that $\mathcal{T} = \mathcal{S}^{\perp}$.*
- *W.l.o.g., $R \in \mathcal{S}$, and $\mathcal{T}$ is induced by an $\mathcal{S}$–filtered tilting module T.*

Filtered modules

Let $\mathcal{S} \subseteq \mathrm{Mod}\text{–}R$. A module M is $\mathcal{S}$–filtered provided there exists a chain of modules and monomorphisms

$$0 = M_0 \overset{\nu_0}{\hookrightarrow} M_1 \overset{\nu_1}{\hookrightarrow} \dots \hookrightarrow M_\alpha \overset{\nu_\alpha}{\hookrightarrow} M_{\alpha+1} \overset{\nu_{\alpha+1}}{\hookrightarrow} \dots \hookrightarrow M_\sigma = M$$

such that

- $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$, and
- for each $\alpha < \sigma$, $\mathrm{Coker}(\nu_\alpha)$ is isomorphic to an element of $\mathcal{S}$.

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Example

Let R be a Dedekind domain. Then the class $\mathcal{D}$ of all divisible modules is 1–tilting (take $\mathcal{S} = \{R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}(R)\}$). $\mathcal{D}$ is induced by the $\mathcal{S}$–filtered tilting module

$$T_{div} = Q \oplus Q/R = Q \oplus \bigoplus_{\mathfrak{p}\in \mathrm{mSpec}(R)} E(R/\mathfrak{p}).$$

The dual setting

Definition

Let R be a ring and $n < \omega$. A left R–module C is n–cotilting provided

(C1) $\mathrm{id}_R(C) \leq n$.

(C2) $\mathrm{Ext}^k_R(C^\kappa, C) = 0$ for all $k \geq 1$ and all cardinals κ.

(C3) There is an exact sequence $0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to W \to 0$, where $C_i \in \mathrm{Prod}\, C$ for each $i \leq n$, and W an injective cogenerator.

The class ${}^\perp C = \{M \in R\text{–Mod} \mid \mathrm{Ext}^k_R(M, C) = 0 \text{ for all } k \geq 1\}$ is the cotilting class induced by C.

The cotilting modules C and C' are equivalent if ${}^\perp C = {}^\perp C'$.

Duality: formal and explicit

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Let R be a commutative ring, $n \geq 0$, and T be an n–tilting R–module. Then the dual module $C = T^* = \mathrm{Hom}_R(T, W)$ is an n–cotilting R–module.

Duality: formal and explicit

Let R be a commutative ring, $n \geq 0$, and T be an n–tilting R–module. Then the dual module $C = T^* = \mathrm{Hom}_R(T, W)$ is an n–cotilting R–module.

Moreover:

If $\mathcal{S} \subseteq \mathrm{mod}–R$ consists of modules of projective dimension $\leq n$ such that $T^\perp = \mathcal{S}^\perp$, then

$$^\perp C = \{N \in R\text{–}\mathrm{Mod} \mid \mathrm{Tor}_k^R(S, N) = 0 \text{ for all } k \geq 1 \text{ and } S \in \mathcal{S}\}$$

is the dual cotilting class induced by C.

If T is $\mathcal{S}$–filtered, then C is $\mathcal{S}^*$–cofiltered.

Cofiltered modules

Let $\mathcal{C} \subseteq \mathrm{Mod}\text{–}R$. A module M is $\mathcal{C}$-cofiltered provided there exists a chain of modules and epimorphisms

$$M = M_\sigma \twoheadrightarrow \dots \overset{\pi_{\alpha+1}}{\twoheadrightarrow} M_{\alpha+1} \overset{\pi_\alpha}{\twoheadrightarrow} M_\alpha \twoheadrightarrow \dots \overset{\pi_1}{\twoheadrightarrow} M_1 \overset{\pi_0}{\twoheadrightarrow} M_0 = 0$$

such that

- $M_\alpha = \varprojlim_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$, and
- for each $\alpha < \sigma$, $\mathrm{Ker}(\pi_\alpha)$ is isomorphic to an element of $\mathcal{C}$.

Example

Let R be a Dedekind domain. Then the class $\mathcal{F}$ of all torsion-free modules is the dual cotilting class induced by the 1-cotilting module

$$C_{tf} = (T_{div})^* \cong Q^\kappa \oplus \prod_{\mathfrak{p} \in \mathrm{mSpec}(R)} J_\mathfrak{p}.$$

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C_{tf} is $\mathcal{C}$-cofiltered, where $\mathcal{C} = \{R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{mSpec}(R)\} \cup \{W\}$, and $W = \bigoplus_{\mathfrak{p} \in \mathrm{mSpec}(R)} E(R/\mathfrak{p})$ is the minimal injective cogenerator.

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Assume R is commutative and noetherian. Then each cotilting module is equivalent to the dual of a tilting one, hence each cotilting class is dual.

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From now on, all rings will be commutative and noetherian.

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There is a 1–1 correspondence between

(i) *the* 1*–cotilting classes* $\mathcal{C}$ *in Mod-R,*

(ii) *the subsets* P *of* $\mathrm{Spec}(R)$ *containing* $\mathrm{Ass}(R)$ *and closed under generalization,*

(iii) *the* 1*–tilting classes* $\mathcal{T}$ *in Mod-R.*

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It is given by the inverse assignments

$\mathcal{C} \mapsto \mathrm{Ass}(\mathcal{C})$ *and* $P \mapsto \{M \in \text{Mod-}R \mid \mathrm{Ass}(M) \subseteq P\}$

and by $P \mapsto \mathcal{T} = \bigcap_{\mathfrak{q} \in \mathrm{Spec}(R) \setminus P} \mathrm{Tr}(R/\mathfrak{q})^{\perp}$

where Tr denotes the Auslander-Bridger transpose.

Characteristic sequences

Definition

A sequence $\mathcal{P} = (P_0, \dots, P_{n-1})$ of subsets of $\mathrm{Spec}(R)$ is called characteristic (of length n in $\mathrm{Spec}(R)$) provided that

(i) P_i is closed under generalization for all $i < n$,

(ii) $P_0 \subseteq P_1 \subseteq \dots \subseteq P_{n-1}$, and

(iii) $\mathrm{Ass}(\Omega^{-i}(R)) \subseteq P_i$ for all $i < n$.

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For each characteristic sequence $\mathcal{P}$, we define the class of modules

$$\mathcal{C}_{\mathcal{P}} = \{M \in \text{Mod-}R \mid \mathrm{Ass}(\Omega^{-i}(M)) \subseteq P_i \text{ for all } i < n\}$$

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Lemma (localization of characteristic sequences)

If $\mathcal{P} = (P_0, \ldots, P_{n-1})$ is characteristic in $\mathrm{Spec}(R)$ and $\mathfrak{m} \in \mathrm{mSpec}(R)$, then the sequence $\mathcal{P}_{\mathfrak{m}} = ((P_0)_{\mathfrak{m}}, \ldots, (P_{n-1})_{\mathfrak{m}})$ is characteristic in $\mathrm{Spec}(R_{\mathfrak{m}})$. Here, $(P_i)_{\mathfrak{m}} = \{\mathfrak{p}_{\mathfrak{m}} \mid \mathfrak{p} \subseteq \mathfrak{m}$ and $\mathfrak{p} \in P_i\}$.

Classification of n–cotilting classes

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Theorem

Let $n \geq 1$, and $\mathcal{P} = (P_0, \dots, P_{n-1})$ be a characteristic sequence. Then $\mathcal{C}_{\mathcal{P}}$ is an n–cotilting class, and the assignments

$$\mathcal{C} \mapsto (\mathrm{Ass}(\mathcal{C}_0), \dots, \mathrm{Ass}(\mathcal{C}_{n-1}))$$

and

$$\mathcal{P} = (P_0, \dots, P_{n-1}) \mapsto \mathcal{C}_{\mathcal{P}}$$

are inverse bijections.

Lemma

Let C be an n–cotilting module with the induced class $\mathcal{C}$. For each $i \leq n$, let $\mathcal{C}_i = {}^{\perp}\Omega^{-i}(C)$. Then $\mathcal{C}_i$ is an $(n-i)$–cotilting class.

A complete classification

Theorem

Let $n \geq 1$. Then there are bijections between:

(i) *the characteristic sequences of length n in $\mathrm{Spec}(R)$,*

(ii) *n–tilting classes $\mathcal{T}$,*

(iii) *n–cotilting classes $\mathcal{C}$.*

A characteristic sequence $(P_0, \dots, P_{n-1})$ corresponds to the n–tilting class

$$\mathcal{T} = \{M \in \mathrm{Mod}\text{–}R \mid Tor_i^R(R/\mathfrak{p}, M) = 0 \, \forall i < n \, \forall \mathfrak{p} \notin P_i\} = \{M \in \mathrm{Mod}\text{–}R \mid Ext_R^1(\mathrm{Tr}(\Omega^{(i)}(R/\mathfrak{p})), M) = 0 \, \forall i < n \, \forall \mathfrak{p} \notin P_i\},$$

and the n–cotilting class

$$\mathcal{C} = \{M \in \mathrm{Mod}\text{–}R \mid Ext_R^i(R/\mathfrak{p}, M) = 0 \, \forall i < n \, \forall \mathfrak{p} \notin P_i\} = \{M \in \mathrm{Mod}\text{–}R \mid Tor_1^R(\mathrm{Tr}(\Omega^i(R/\mathfrak{p})), M) = 0 \, \forall i < n \, \forall \mathfrak{p} \notin P_i\}.$$

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Example

Let $C = \bigoplus_{\mathfrak{m} \in \mathrm{mSpec}(R)} E(R/\mathfrak{m})$. Then C is a minimal 0-cotilting module (= minimal injective cogenerator).

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Theorem (existence)

Let $\mathcal{C}$ be an n–cotilting class. Then there is a minimal n-cotilting module C inducing $\mathcal{C}$.

Compatible families of characteristic sequences

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For $\mathfrak{p} \in \operatorname{Spec}(R_\mathfrak{m})$, let $\widehat{\mathfrak{p}}$ denote the prime ideal of R such that $\widehat{\mathfrak{p}} \subseteq \mathfrak{m}$ and $(\widehat{\mathfrak{p}})_\mathfrak{m} = \mathfrak{p}$. Similarly, we define $\widehat{P}$ for $P \subseteq \operatorname{Spec}(R_\mathfrak{m})$.

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Definition

Let $n < \omega$. Then $\mathfrak{P}$ is a compatible family of characteristic sequences provided $\mathfrak{P} = (\mathcal{P}(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{mSpec}(R))$, where for each $\mathfrak{m} \in \mathrm{mSpec}(R)$,

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- $\mathcal{P}(\mathfrak{m}) = (P_{0,\mathfrak{m}}, \dots, P_{n-1,\mathfrak{m}})$ is a characteristic sequence in $\mathrm{Spec}(R_{\mathfrak{m}})$, and
- $\widehat{P_{i,\mathfrak{m}}}$ and $\widehat{P_{i,\mathfrak{m}'}}$ contain the same prime ideals from the set $\{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \subseteq \mathfrak{m} \cap \mathfrak{m}'\}$, for all $\mathfrak{m}, \mathfrak{m}' \in \mathrm{mSpec}(R)$ and $i < n$.

Tilting and localization

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Theorem

- *Let T be an n–tilting R–module with characteristic sequence $\mathcal{P}$. Then for each $\mathfrak{m} \in \mathrm{mSpec}(R)$, $T_{\mathfrak{m}}$ is an n–tilting $R_{\mathfrak{m}}$-module with characteristic sequence $\mathcal{P}_{\mathfrak{m}}$.*

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  Moreover, $\mathfrak{P} = (\mathcal{P}_\mathfrak{m} \mid \mathfrak{m} \in \mathrm{mSpec}(R))$ is a compatible family of characteristic sequences.
- *Conversely, assume that for each $\mathfrak{m} \in \mathrm{mSpec}(R)$, $T(\mathfrak{m})$ is an n–tilting $R_\mathfrak{m}$-module with characteristic sequence $\mathcal{P}(\mathfrak{m})$, and the family $(\mathcal{P}(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{mSpec}(R))$ is compatible.*

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  Then there exists a unique tilting R–module T such that $T_\mathfrak{m}$ is equivalent to $T(\mathfrak{m})$ for each $\mathfrak{m} \in \mathrm{mSpec}(R)$.

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  Then there exists a unique tilting R–module T such that $T_{\mathfrak{m}}$ is equivalent to $T(\mathfrak{m})$ for each $\mathfrak{m} \in \mathrm{mSpec}(R)$.

Open problem

How to recover T from the family $(T(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{mSpec}(R))$?

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Properties

- Unlike localization, the colocalization is only left exact in general.
- Colocalization commutes with inverse limits, and it is exact on short exact sequences with pure–injective kernels.
- The colocalization of an $\mathcal{S}^*$-cofiltered cotilting R–module is an $(\mathcal{S}^*)^{\mathfrak{m}}$–cofiltered cotilting $R_{\mathfrak{m}}$–module, and $(\mathcal{S}^*)^{\mathfrak{m}} = (\mathcal{S}_{\mathfrak{m}})^*$.

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  (Recall that $P_i = \mathrm{Ass}(^{\perp}\Omega^{-i}(C))$ for all $i < n$.)

  Then for each $\mathfrak{m} \in \mathrm{mSpec}(R)$, $C^{\mathfrak{m}}$ is an n–cotilting $R_{\mathfrak{m}}$-module with characteristic sequence $\mathcal{P}_{\mathfrak{m}}$.
  Moreover, $\mathfrak{P} = (\mathcal{P}_{\mathfrak{m}} \mid \mathfrak{m} \in \mathrm{mSpec}(R))$ is a compatible family of characteristic sequences.

- *Conversely, assume that for each $\mathfrak{m} \in \mathrm{mSpec}(R)$, $C(\mathfrak{m})$ is an n–cotilting $R_{\mathfrak{m}}$-module with characteristic sequence $\mathcal{P}(\mathfrak{m})$, and the family $(\mathcal{P}(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{mSpec}(R))$ is compatible.*
  Then there is a unique cotilting R–module C such that $C^{\mathfrak{m}}$ is equivalent to $C(\mathfrak{m})$ for each $\mathfrak{m} \in \mathrm{mSpec}(R)$.

Cotilting and colocalization

Theorem

- *Let C be an n–cotilting R–module with characteristic sequence $\mathcal{P}$.*

  (Recall that $P_i = \mathrm{Ass}({}^{\perp}\Omega^{-i}(C))$ for all $i < n$.)

  Then for each $\mathfrak{m} \in \mathrm{mSpec}(R)$, $C^{\mathfrak{m}}$ is an n–cotilting $R_{\mathfrak{m}}$-module with characteristic sequence $\mathcal{P}_{\mathfrak{m}}$.
  Moreover, $\mathfrak{P} = (\mathcal{P}_{\mathfrak{m}} \mid \mathfrak{m} \in \mathrm{mSpec}(R))$ is a compatible family of characteristic sequences.

- *Conversely, assume that for each $\mathfrak{m} \in \mathrm{mSpec}(R)$, $C(\mathfrak{m})$ is an n–cotilting $R_{\mathfrak{m}}$-module with characteristic sequence $\mathcal{P}(\mathfrak{m})$, and the family $(\mathcal{P}(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{mSpec}(R))$ is compatible.*
  Then there is a unique cotilting R–module C such that $C^{\mathfrak{m}}$ is equivalent to $C(\mathfrak{m})$ for each $\mathfrak{m} \in \mathrm{mSpec}(R)$.

  Moreover, we can easily recover C as $C = \prod_{\mathfrak{m} \in \mathrm{mSpec}(R)} C(\mathfrak{m})$.

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