Some Model Theory of Modules over Bézout Domains

Carlo Toffalori (Camerino)

ASTA, Spineto, June 16–20, 2014

A joint work with Gena Puninski (Minsk) ...
... and our common tribute to Alberto

A commutative domain B with identity is *Bézout* if every 2-generated ideal (hence every finitely generated ideal) is principal.

A Bézout domain is *coherent*: the intersection of 2 principal ideals is also principal.

Then can define, for every $a, b \in B$,

- a greatest common divisor $\gcd(a, b)$,
- a least common multiple $\mathrm{lcm}(a, b)$

satisfying the Bézout identities.

Bézout domains are Prüfer: their localizations at maximal ideals are commutative valuation domains.

Bézout domains include

- the ring of algebraic integers,
- the ring of entire (complex or real) functions in 1 variable,
- $\mathbb{Z} + X\mathbb{Q}[X]$,
- more generally the rings coming from the so-called $D + M$-construction:
  - take a *PID* D which is not a field,
  - consider its field of fractions Q,
  - form $B = D + XQ[X]$
  - and get a Bézout domain, which is neither Noetherian nor a *UFD* (because for every prime $p \in D$

    $$XB \subset p^{-1}XB \subset p^{-2}XB \subset \dots$$

    is a strictly ascending chain of ideals).

Our interest. The model theory of (right) modules over a Bézout domain B.

To explore

- pp-formulas and pp-types over B
- the Ziegler spectrum $\mathrm{Zg}(B)$ (where
  - points = (isomorphism types of) indecomposable pure injective B-modules,
  - basic open sets = $(\varphi(x)/\psi(x)) :=$ $\{N \in Zg(B) : \varphi(N) \supset \psi(N) \cap \varphi(N)\}$ where $\varphi(x)$, $\psi(x)$ range over pp-formulas).

Some possible aims

- Try to classify indecomposable pure injective B-modules, or
- show existence of pathologies like superdecomposable pure injective B-modules.

Useful to consider $\Gamma^+(B)$ = lattice of principal ideals of B with respect to divisibility of generators (so with respect to reverse inclusion).

Theorem

Every pp-formula in 1 variable over B is logically equivalent to

1. *a finite conjunction of formulas* $\varphi_{a,b}(x) : a \mid x + xb = 0$ *with* $a, b \in B$*, or also to*
2. *a finite sum of formulas* $\psi_{a,b}(x) : a \mid x \wedge xb = 0$ *with* $a, b \in B$*.*

Note: An effective reduction can be obtained over an effectively given B.

Now have to understand logical equivalence, hence logical implication, between formulas $\varphi_{a,b}$. Here is an algebraic characterization.

Theorem

For $a, b, c, d \in B$ and $a, d \neq 0$, $\varphi_{a,b} \to \varphi_{c,d}$ if and only if $c \mid a$ and c, $\mathrm{lcm}(b, d)/c$ are coprime.

Theorem

A basis of open sets of $Zg(B)$ is given by $(\psi_{c,d}/\varphi_{a,b})$ where a, b, c, d range over B.

Note: The sets in this basis an be effectively enumerated over an effectively given B.

How to characterize (indecomposable) pp-types (and points in the spectrum).

Theorem

There is a natural 1-1 correspondence between

- *indecomposable pp-types q (in 1 variable) over B,*
- *admissible pairs (I, J) of filter/cofilter partitions of $\Gamma^+(B)$*

sending q to

- $I = \{bB \in \Gamma^+(B) : xb = 0 \in q\}$ *(a filter),* $I' = \Gamma^+(B) \setminus I$,
- $J = \{aB \in \Gamma^+(B) : a \mid x \notin q\}$ *(a cofilter),* $J' = \Gamma^+(B) \setminus J$.

Admissibility: if $aB \in J'$, $cB \in J$, $bB \in I$, $dB \in I'$, a properly divides c and d properly divides b, then c/a and b/d are coprime.

Theorem

There is a natural 1-1 correspondence between

- *pp-types q (in 1 variable) over B,*
- *functions F from $\Gamma^+(B)$ to the set of cofilters of $\Gamma^+(B)$ – a lattice with respect to inclusion, and indeed the completion of $\Gamma^+(B)$ – satisfying the conditions 1–5 below. For every $a, b \in B$, one puts $\varphi_{a,b} \in q$ if and only if $aB \in F(bB)$.*

The five conditions:

1. $F(0B) = \Gamma^+(B)$
2. $F(B) = \Gamma^+(B)$ *if and only if* $q = 0$
3. *For every $a, b, b' \in B$, if $aB \in F(bB)$, then $ab'B \in F(bb'B)$, in particular F is non-decreasing*
4. *For every $a, b, b' \in B$, if $aB \in F(bb'B)$ and a, b' are coprime, then $aB \in F(bB)$*
5. *F preserves meet, that is, for every $b, b' \in B$, $F(\gcd(b, b')B) = F(bB) \cap F(b'B)$.*

On this basis...
superdecomposable modules sometimes occur over B.

Recall that a non–zero (pure injective) B-module M is *superdecomposable* if and only if it does not admit no non-zero indecomposable summand.

Width and superdecomposables (Ziegler). Let R be any ring.

- If there is a superdecomposable pure injective R-module, then the lattice of pp-formulas over R has no width.
- The converse is also true when the lattice is countable (open in general).

Coming back to Bézout domains

Theorem
If B has a non-zero proper idempotent ideal, then B possesses a superdecomposable pure injective module.

It applies to

- the ring of algebraic integers,
- the ring of entire (real or complex) functions in 1 variable.

Theorem

The width of B is undefined if and only if the value group of B contains a densely ordered subchain.

A detailed analysis of this case generalizes that over commutative valuation domains (actually rings) using the characterization of pp-types via $\Gamma^{+}(B)$ and the associated functions F.

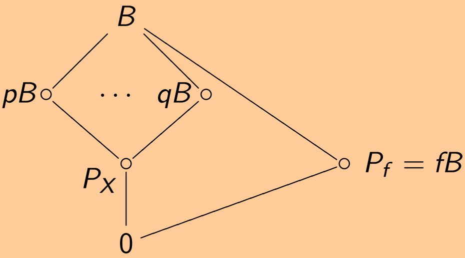
The $D+M$–construction case, $B = D + XQ[X]$

Lemma

A nonzero prime ideal of B is either

1. *pB, where p is a prime element of D, or*
2. *for some irreducible polynomial $f(X) \in Q[X]$ whose constant term is 1, the ideal $P_f = f(X)B$, or*
3. *$P_X = XQ[X]$.*

These prime ideals satisfy the following inclusion diagram:



In particular, P_X is not maximal and $P_X = \cap_p pB$.

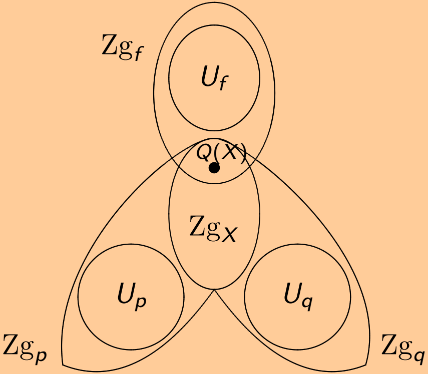
Theorem

The Krull–Gabriel dimension of B equals 4 with $Q(X)$ being a unique point of maximal CB-rank. Moreover the Ziegler spectrum $\mathrm{Zg}(B)$ of B is the union of the closed subspaces

- $\mathrm{Zg}(B_{f(X)})$ *where $f(X)$ ranges over the irreducible polynomials of $Q[X]$ with constant term 1,*
- $\mathrm{Zg}(B_p)$ *where p ranges over the prime elements of D.*

The latter subspaces include $\mathrm{Zg}(B_X)$, which is their intersection. The intersection of two different $\mathrm{Zg}(B_{f(X)})$, or of a $\mathrm{Zg}(B_{f(X)})$ and a $\mathrm{Zg}(B_p)$, reduces to the only point $Q(X)$.

The picture – a sort of bouquet ...
Here Zg_f, Zg_p and Zg_X abbreviate $\mathrm{Zg}(B_f)$, $\mathrm{Zg}(B_p)$, $\mathrm{Zg}(B_X)$ respectively.



U_f, U_p are the open sets where f, p respectively act as non-isomorphisms.

Decidability

A countable integral domain D is said to be effectively given if its elements can be recursively listed (possibly with repetitions) as

$$r_0 = 0, r_1 = 1, r_2, \ldots, r_k, \ldots \qquad k \in \mathbb{N}$$

so that the following holds:

- there are algorithms which, given $n, m \in \mathbb{N}$, produce $r_n + r_m$, $-r_n$ and $r_n \cdot r_m$ (more precisely indices for these elements in the list);
- there is an algorithm which, given $n, m \in \mathbb{N}$, decides whether $r_n = r_m$ or not;
- there is an algorithm which, given $n, m \in \mathbb{N}$, establishes whether $r_m \mid r_n$ or not.

We say that a(n effectively given) principal ideal domain D is *strongly effectively given* if it satisfies the following extra conditions:

- there is an algorithm that lists all the prime elements of D;
- there is an algorithm that lists all the irreducible polynomials of $Q[X]$;
- for every prime p the size of the field D/pD is known.

For instance $\mathbb{Z}$ is strongly effectively given (Kronecker).

Theorem

Let D be a strongly effectively given principal ideal domain and let $B = D + XQ[X]$ be the corresponding Bézout domain. Then the theory $T(B)$ of B-modules is decidable.

The proof uses the previous analysis and Lorna Gregory's work on decidability of modules over valuation domains.

- G. Puninski – C. T., Some model theory of modules over Bézout domains. The width, *J. Pure Applied Algebra*, to appear
- G. Puninski – C. T., Decidability of modules over a Bézout domain $D + XQ[X]$ with D a principal ideal domain and Q its field of fractions, *J. Symbolic Logic*, to appear