Grothendieck topologies and weak limits*

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The construction of the elementary quotient completion of an elementary doctrine is an excellent tool to produce models of constructive theories for mathematics, see [10, 11]. In particular, they offer crucial models for the Minimalist Foundation proposed in [12, 9] and allows for a careful analysis of the setoid models obtained from Martin-Löf Type Theory, see *e.g.* [14].

The elementary quotient completion extends the well-known categorical construction of the exact completion $\mathcal{A}_{\mathrm{ex/wlex}}$ of a given category $\mathcal{A}$ with weak limits provided that products are strong, see [4, 6].

In fact, connections between the exact completion of a category with weak finite limits with constructive mathematics had already been presented in several occasions, see *e.g.* [3, 1, 2], as several elementary toposes are exact completions, see [13] which contains also a characterisation of those exact completions which are elementary toposes. There are also characterisations of those exact completions which are cartesian closed categories in [15], and of those which are locally cartesian closed categories [5]. But these characterisation always invoke that the given category $\mathcal{A}$ has strong finite products.

This matches with the situation of an elementary doctrine $P\colon \mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{P}os$ where the base category $\mathcal{B}$ is required to have finite products. More precisely, recall that an ***elementary doctrine*** is a functor $P\colon \mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{P}os$ such that $\mathcal{B}$ has finite limits, all fibres $P(b)$ are inf-semilattices, for each arrow $f\colon b \longrightarrow b'$ in $\mathcal{B}$ the reindexing functors $P(f)\colon P(b') \longrightarrow P(b)$ preserves finite infs, and for every object b in $\mathcal{B}$ there is an object δ_b in $P(b \times b)$ such that

(a) $\top \leq P(\langle \mathrm{id}_b, \mathrm{id}_b\rangle)(\delta_b)$;

(b) $P(\mathrm{pr}_1)(\alpha) \wedge \delta_b \leq P(\mathrm{pr}_2)(\alpha)$
for every α in $P(b)$ where $\mathrm{pr}_i\colon b \times b \longrightarrow b$, $i = 1, 2$, are the two projections;

(c) $P(\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle)(\delta_{b_1}) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_4\rangle)(\delta_{b_2}) \leq \delta_{b\times c}$
where $\mathrm{pr}_1, \mathrm{pr}_3\colon b_1 \times b_2 \times b_1 \times b_2 \longrightarrow b_1$, $\mathrm{pr}_2, \mathrm{pr}_4\colon b_1 \times b_2 \times b_1 \times b_2 \longrightarrow b_2$ are the four projections.

There are two families of examples of elementary doctrines, one from logic, the other from category theory, see [10] for details. For the first, consider a theory $\mathcal{T}$ in first order logic: the base category $\mathcal{B}$ is the so-called category of contexts and substitutions of (the language of) $\mathcal{T}$: objects are lists of distinct variables, arrows are lists of terms in the variables of the domain and in number equal to that of the variables in the codomain. Composition is given by substitution. The fibre on the context $\langle x_{i_1}, \ldots, x_{i_n}\rangle$ is the Lindenbaum-Tarski algebra of well-formed formulas in $\mathcal{T}$ with the free variables among $\langle x_{i_1}, \ldots, x_{i_n}\rangle$, with respect to derivability in $\mathcal{T}$. Reindexing is given by substitution. The object $\delta_{\langle x_{i_1},\ldots,x_{i_n}\rangle}$ is the (equivalence class of the) conjuction of equality predicates $\bigwedge_k (x_{i_k} = x'_{i_k})$.

For the second family of examples, consider a category $\mathcal{C}$ with (strict) finite products and with weak equalisers. The base category of the doctrine is $\mathcal{C}$ itself. The fibre $P(c)$ is the poset reflection of the comma category $\mathcal{C}/c$. Reindexing is given by weak pullback.[1] The object δ_c is the (equivalence class of the) diagonal $\langle \mathrm{id}_c, \mathrm{id}_c\rangle\colon c \longrightarrow c \times c$.

The elementary quotient completion of the elementary doctrine $P\colon \mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{P}os$ has objects in the base category which are formal equivalence relations in the logic of P with arrows which are those in $\mathcal{B}$ which preserve the formal equivalence relation. A fibre consists of descent data with respect to the formal equivalence relation, see [10] for details.

The peculiarity of strong finite products with respect to weak limits is certainly apparent. Recent work [8] by one of the collaborators of the present project produced a solution for the general characterisation of those exact completions $\mathcal{A}_{\mathrm{ex/wlex}}$ which are locally cartesian closed. In the work [7] for his PhD thesis, another of the collaborators determined a suitable

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[1] Weak limits in $\mathcal{C}$ ensure weak limits in the comma category $\mathcal{C}/c$. And these are enough to ensure that the poset reflection of $\mathcal{C}/c$ has (strong) limits, *i.e.* finite infs, and that reindexing is functorial.

set of conditions to present an extension of the notion of elementary doctrine with respect to a base category $\mathcal{B}$ with just weak finite products. It requires that the δ's behave with some kind of bias with respect to a specific weak product diagram—hence the name ***biased elementary doctrine***—and showed how to produce an elementary doctrine from a biased elementary doctrine, and how the elementary quotient completion extends to the wider settings as a 2-categorical left adjoint.

We show how the two extensions refer to the same situation which involves the product completion $\mathcal{A}_{\mathrm{pr}} := (\mathrm{Fam_{fin}}(\mathcal{A}^{\mathrm{op}}))^{\mathrm{op}}$ of a category $\mathcal{A}$. When $\mathcal{A}$ has weak finite limits there is a Grothendieck topology Θ where covers contain a diagram of weak binary products.

Note that the embedding $D\colon \mathcal{A} \hookrightarrow \mathcal{A}_{\mathrm{pr}}$ covers with respect to the Grothendieck topology, in the sense that for every object in $\mathcal{A}_{\mathrm{pr}}$ there is a coverage of arrows whose domains are all in the image of the embedding. Note also that since in general the 2-category of $\mathcal{B}$-indexed posets is equivalent to the 2-category $\mathbf{Pos}([\mathcal{B}^{\mathrm{op}}, \mathbf{Set}])$, every $\mathcal{A}_{\mathrm{pr}}$-indexed poset can be completed to a Θ-sheaf.

Theorem 1. *Let $\mathcal{A}$ be a category with weak limits. Let $P\colon {\mathcal{A}_{\mathrm{pr}}}^{op} \longrightarrow \mathbf{Pos}$ be a doctrine which is a Θ-sheaf. The following are equivalent:*

(i) *The doctrine $P\colon {\mathcal{A}_{\mathrm{pr}}}^{op} \longrightarrow \mathbf{Pos}$ is elementary.*

(ii) *The doctrine $P\restriction_{\mathcal{A}^{op}}\colon \mathcal{A}^{op} \longrightarrow \mathbf{Pos}$ is biased.*

Theorem 2. *Let $\mathcal{A}$ be a category with weak limits. There is a full embedding $\mathcal{A}_{\mathrm{ex/wlex}} \hookrightarrow \mathrm{sh}(\mathcal{A}_{\mathrm{pr}}, \Theta)$ of the exact completion in the category of Θ-sheaves which is exact and preserves any local exponential which exists in $\mathcal{A}_{\mathrm{ex/wlex}}$.*

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