ON THE COMPUTATIONAL PROPERTIES OF BASIC MATHEMATICAL NOTIONS

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Abstract. We investigate the computational properties of basic mathematical notions pertaining to $\mathbb{R} \rightarrow \mathbb{R}$-functions and subsets of $\mathbb{R}$, like *finiteness*, *countability*, (*absolute*) *continuity*, *bounded variation*, *suprema*, and *regularity*. We work in higher-order computability theory based on Kleene's S1-S9 schemes. We show that the aforementioned italicised properties give rise to a number of robust classes of computationally equivalent operations, the latter based on well-known theorems from the mainstream mathematics literature, including the uncountability of $\mathbb{R}$.

1. Overview

Our starting point is provided by the following most basic questions.

Given a finite set, *how many* elements does it have has, and *which ones*?

We study these and related questions in Kleene's *higher-order computability theory*, based on his computation schemes S1-S9 ([4, 5]). In particular, a central object of study is the higher-order functional Ω which on input a finite set of real numbers, list the elements as a finite sequence.

First of all, the 'finiteness' functional Ω give rise to a *huge and robust* class of computationally equivalent operations, called the Ω-cluster, as sketched in Section 2. For instance, many basic operations on functions of *bounded variation* are part of the Ω-cluster, including those stemming from the well-known *Jordan decomposition theorem.* In addition, we identify a second cluster of computationally equivalent objects, called the Ω_1-cluster, based on Ω_1, the restriction of Ω to singletons. We also show that both clusters include basic operations on *regulated* and *Sobolev space* functions, respectively a well-known super- and sub-class of the class of BV-functions.

Secondly, the uncountability of $\mathbb{R}$ suggests a natural computational task:

given a countable set $A \subset \mathbb{R}$, find $x \in \mathbb{R} \setminus A$.

Any functional that performs this operation is called a *Cantor realiser*, with appropriate modifiers (weak, intermediate, strong) depending on the representation of the countable set. Cantor realisers give rise to at least two clusters, including functionals stemming from Volterra's early work, basic properties of the Riemann integral, and Blumberg's theorem, as discussed in Section 3.

Finally, initial results have been published as [6, 8] while an overview of the above may be found in [7].

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2. Two robust clusters

The following theorems give rise to functionals in the Ω and Ω_1-clusters.

- The Jordan decomposition theorem (Jordan, 1881 [3]).
- A function of bounded variation on the unit interval has a supremum.
- For a function of bounded variation on the unit interval, there is a sequence listing all points of discontinuity.
- A countable set (=injection or bijection to $\mathbb{N}$) can be enumerated.
- A regulated function on the unit interval has a supremum.
- For a regulated function on the unit interval, there is a sequence listing all points of discontinuity.
- For regulated $f : [0,1] \to \mathbb{R}$, there are g, h such that $f = g \circ h$ with g continuous and h strictly increasing on their domains (Sierpiński, [9]).
- For a regulated function, the Banach indicatrix exists.
- A regulated function is of class Baire 1.
- A regulated upper semi-continuous function on $[0,1]$ attains a maximum

This list is by no means exhaustive, as is clear from [7, §4].

3. Two or more robust clusters

The centred operation from Section 1 gives rise to numerous computational equivalences, based on the following theorems.

- For regulated $f : [0,1] \to \mathbb{R}$, there is a point $x \in [0,1]$ where f is continuous (or quasi-continuous, or lower semi-continuous, or Darboux).
- For regulated $f : [0,1] \to [0,1]$ with Riemann integral $\int_0^1 f(x)dx = 0$, there is $x \in [0,1]$ with $f(x) = 0$ (Bourbaki, [2, p. 61, Cor. 1]).
- (Volterra [10]) For regulated $f, g : [0,1] \to \mathbb{R}$, there is $x \in [0,1]$ such that f and g are both continuous or both discontinuous at x.
- (Volterra [10]) For regulated $f : [0,1] \to \mathbb{R}$, there is either $q \in \mathbb{Q} \cap [0,1]$ where f is discontinuous, or $x \in [0,1] \setminus \mathbb{Q}$ where f is continuous.
- For regulated $f : [0,1] \to \mathbb{R}$, there is $y \in (0,1)$ where $F(x) := \lambda x. \int_0^x f(t)dt$ is differentiable with derivative equal to $f(y)$.
- For regulated $f : [0,1] \to \mathbb{R}$, there are $a, b \in [0,1]$ such that $\{x \in [0,1] : f(a) \leq f(x) \leq f(b)\}$ is infinite.
- Blumberg's theorem ([1]) restricted to regulated functions.

This list is by no means exhaustive, as is clear from [8, §3]. We discuss similar results for Baire 1 functions and the associated *Baire category theorem.*

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