IMPLICATIVE ALGEBRAS FOR MODIFIED REALIZABILITY

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One of the aims of this talk is to present a topos for modified realizability. Modified realizability is an interpretation of intuitionistic arithmetic in finite types presented by Kreisel [3, 4]. Its original motivation was to present an interpretation of this system which would refute Markov's Principle.

Modified realizability can be characterised – in the sense of Troelstra – by two principles: the axiom of choice for all finite types and the independence of premise principle [8]. This means two things: first of all, these principles can be interpreted without assuming them in the metatheory. Secondly, if one assumes these principles one can show that any formula is equivalent to it being modified realized.

In the categorical tradition various toposes have been presented as semantic analogues of proof-theoretic interpretations. For example, the effective topos is the topos for Kleene's number realizability ([1]; see [6] for more on realizability toposes). In the literature one can find a topos for modified realizability (originally due to Grayson), but in it the axiom of choice for all finite types fails [7, 6]. One aim of this talk is to present another modified realizability topos in which both the axiom of choice for all finite type and the independence of premise principle hold. I will argue that for this reason it has a stronger claim to deserve the title "modified realizability topos". (This is based on joint work with Mees de Vries, who presented this topos in his MSc thesis written under my supervision.)

In this talk I will present this topos via Alexandre Miquel's notion of an "implicative algebra" [5]. This notion of an implicative algebra generalises both forcing and realizability and can be used to present any topos which derives from a tripos. In this talk I will explain how both Grayson's and the improved modified realizability topos can be obtained by performing a certain construction on an implicative algebra: Grayson starts from the implicative algebra for the effective topos (the powerset of the natural numbers, essentially), while the construction of De Vries and myself does the same construction but starting from the implicative algebra for extensional realizability (the set of all PERs, essentially).

For this to work nicely it is important to develop the theory of nuclei (*aka* local operator, or Lawvere-Tierney topologies) for implicative algebras. I will give a brief account of this theory. If time permits, I will also sketch how a topos for Ulrich

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Date: August 23, 2022.

Kohlenbach's "monotone modified realizability" [2] can be obtained by these methods as well.

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